International Journal of Business Marketing and Management (IJBMM)
Volume 6 Issue 4 April 2021, P.P. 15-27
ISSN: 2456-4559
www.ijbmm.com

Optimal investment strategy based on semi-variable transaction costs

Shifeng Shen[1], PeibiaoZhao[2]
[1](School of Science, Nanjing University of Science and Technology, China)
[2](School of Science, Nanjing University of Science and Technology, China)

Abstract: *In this paper, we focus on a financial market with one riskless and one risky asset, and consider the asset allocation problem in the form of semi-variable transaction costs. One of the basic ideas of this paper is to transform the problem of maximizing the expected utility of terminal wealth in a friction market with semi-variable transaction costs into a frictionless market which can produce the same maximum utility, and then give the analytical formula for the original problem. Generally, the price process of risky assets in such frictionless market is called "shadow price", and the corresponding problem after conversion is called "shadow problem".*
Keywords –*Utility maximization, Semi-variable cost, Shadow price*

I. Introduction

In the field of financial mathematics, asset allocation is a very hot and difficult problem. Generally speaking, investors invest their money in different types of assets in financial market in order to maximize the expected utility of their wealth at some point in the future. This kind of problem is usually regarded as the utility maximization problem with constraints.

A complete financial market is a kind of very ideal investment environment, where any assets in this market can be bought and sold without transaction costs. In a complete financial market, no arbitrage is equivalent to the existence of a unique equivalent martingale measure Q, such that the security price under this measure is a martingale, and this property can be used to price a contingent claim. At the same time, all assets can be replicated with the underlying assets. However, the assumption of the complete financial market is not consistent with the real investment environment faced by investors. Therefore, it is of more practical significance to consider the optimization of the portfolio in the case of incomplete market.

Investment in incomplete market is faced with transaction costs. Specifically, investors buy and sell assets in different prices. The ask price is higher, while the bid price is lower because of transaction costs. Due to the existence of transaction costs, investors have to balance between transaction profits and payments.

The research on incomplete financial market still started from the research on complete market. By applying convex analysis and martingale properties, Pliska[1] solved the problem of maximizing the expected utility of wealth at a terminal planning horizon by selecting portfolio of securities. Karatzas et al.[2] studied that when the number of stocks is less than the dimension of multi-dimensional Brownian motion, the incomplete market can be transformed into a complete market by introducing "virtual" stocks, and they proved that the optimal portfolio obtained in this method is consistent with that in the original incomplete market. Kramkov and Schachermayer[3] studied the problem of maximizing the expected utility of terminal wealth in the framework of a general incomplete semi-martingale model of a financial market. They showed that the necessary and sufficient condition on a utility function for the validity of the theory to hold true is that asymptotic elasticity of the utility function should be strictly less than 1.

Liu and Loewenstein[4] studied the optimal trading strategy for a CRRA investor who wants to maximize the expected utility of wealth on a finite date when facing transaction costs. They showed that even small transaction costs can have large impact on the optimal portfolio. Hence, an interesting question is that if this impact can be replaced in a frictionless market which yields the same optimal strategy and utility. If so, this frictionless market is called “shadow market”. The concept of such shadow market was first proposed by Cvitanić and Karatzas[5]. In their pioneering work, they found that if a dual problem was solved by a suitable solution, then the optimal portfolio is the one that hedges the inverse of marginal utility evaluated at the shadow price density solving the corresponding dual problem. Later, Kallsen and Muhle-Karbe[6] found that shadow price always exists in finite space. Benedettiet al.[7] showed that if short selling is not allowed in the financial market, then shadow price can always exist. As for càdlàg security price process S, Czichowsky and Schachermayer[8] proved that shadow price can be defined by means of a “sandwiched” process which consists of a predictable and an optional strong super-martingale. This conclusion was then extended by Bayraktar and Yu[9] to a similar problem with random endowment.

Instead of constructing the shadow price from dual problem, Loewenstein[10] assumed that short sales was not allowed and investor faced transactions costs, then he proved that shadow price can be constructed from the derivatives of dynamic primal value functions. This result was then extended by Benedetti and Campi[7] to a similar problem with Kabanov’s muti-currency model.

In addition, shadow price plays an important role in optimization. Under the geometric Brownian motion model, the optimal investment and consumption problem with logarithmic utility function is studied by Kallsen and Muhle-Karbe[11] using the results of stochastic control theory, then shadow price was constructed by solving the free boundary problem. Some researchers have also studied in the form of logarithmic utility function[12] and power utility function[13].

In view of establishing and solving the utility maximization model simply, most of researches on shadow market focus on the assumption that there are only proportional transaction costs for the trade of risky assets in financial market. Most of the changes in the research only focus on the form of utility function, the form of the price process for risky assets, and the description for the financial market. Although the hypothesis of proportional transaction costs can be used to prove the existence of the solution to the utility maximization problem and to establish the duality problem simply, there is obviously a big difference between the hypothesis and the trading market in our real life. Therefore, it is of great significance to extend the proportional transaction costs to match our real market.

The remainder of this article goes as follows. In Section 2 we formulate the utility maximization problem with semi-variable cost and prove the existence of its solution. In Section 3 we first present that shadow price can be constructed under semi-variable cost, then give the recurrence formula of the optimal strategy under the friction market. Section 4 is a case analysis in order to prove our conclusion.

II. Utility maximization problem with semi-variable cost

This section is to consider the problem of securities investment with semi variable cost. By using the basic theory of stochastic process, especially the properties and conclusions of martingale method, we prove that there is a unique optimal trading strategy for the problem of maximizing the expected utility of terminal wealth in this friction market through the estimation of total variation of self financing trading strategy.

2.1 Construction of the wealth expected utility maximization model

For the sake of simplicity, we consider a market only consisting of a riskless asset and a risky asset. The riskless asset has a constant price 1, and the trade of risky asset needs transaction costs. For example, an investor needs to pay a higher ask price S when buying, but only receives a lower bid price $[(1-\lambda)S-C]$ when selling. Here $\lambda \in (0,1)$ is proportional transaction cost rate, and C is a constant on behalf of the commission every time the investor trade assets in market.

Assumption 2.1.1 The price of risky asset $S=(S_t)_{0\le t\le T}$ is positive and Right Continuous with Left Limits, and is adapted to probability space$(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},\mathbb{P})$. Moreover,$\mathcal{F}_{t-}=\mathcal{F}_tandS_{T-}=S_T$.

Remark 2.1.2 Throughout this paper, we assume that the price of risky asset S cannot jump at terminal time T, thus we have $\mathcal{F}_{t-}=\mathcal{F}_tandS_{T-}=S_T$, which means that the investor can also liquidate his position in risky assets at terminal time T.

Definition 2.1.3 Trading strategy with semi-variable cost is an R^2-valued, predictable, finite variation process$\varphi=(\varphi_t^0,\varphi_t^1)_{0\le t\le T}$, where φ_t^0 describes the holdings in the riskless asset and φ_t^1 describes the holdings in the risky asset.

During the whole paper, there is no additional capital in the process of investment except for initial investment, and no capital is moved out of the market. Thus the following definition of self-financing trading strategy can be well-defined:

Definition 2.1.4 A finite variation process trading strategy $\varphi = (\varphi_t^0, \varphi_t^1)_{0\le t\le T}$ for all $0 \le s \le t \le T$ is called self-financing trading strategy which satisfies

$$\int_s^t d\varphi_u^{0,+} = \int_s^t [(1-\lambda)S_u - c]d\varphi_u^{1,-}, \quad \int_s^t d\varphi_u^{0,-} = \int_s^t S_u d\varphi_u^{1,+}, \qquad (1)$$

where $d\varphi_u^{0,+}$ and $d\varphi_u^{1,+}$ denote the holding of increase and decrease in riskless asset by investors, while $d\varphi_u^{0,-}$ and $d\varphi_u^{1,-}$ denote the holding of increase and decrease in risky asset.

In addition, we assume that every time the investor in this market cannot short any position, then we consider a utility maximization problem in this constraint.

Assumption 2.1.5 Suppose that the investor's preferences are modeled by a standard utility function $U : \mathbb{R}_+ \to \mathbb{R}$, which also satisfy the Inada conditions

$$U'(0) = \lim_{x\to 0} U'(x) = \infty \text{ and } U'(\infty) = \lim_{x\to\infty} U'(x) = 0. \qquad (2)$$

Assumption 2.1.6 The utility function U satisfies the Reasonable Asymptotic Elasticity, i.e.

$$AE(U) := \lim_{x\to\infty} \sup \pi(x) = \lim_{x\to\infty} \sup \frac{xU'(x)}{U(x)} < 1 \qquad (3)$$

Remark 2.1.7 For the utility function U, $AE(U) < 1$ is a necessary and sufficient condition for the existence of the optimal trading strategy in the problem (4). In practical sense, the elastic function $\pi(x)$ represents the ratio of marginal utility $U'(x)$ to average utility $U(x) / x$.

Then the problem faced by an investor is to find an optimal trading strategy $\hat{\varphi} = (\hat{\varphi}_t^0, \hat{\varphi}_t^1)$ to maximize the expected utility of terminal wealth

$$\max \mathbb{E}[U(\varphi_T^0 + \varphi_T^1[(1-\lambda)S_T - c])], \varphi \in \mathcal{A}(x) \qquad (4)$$

where $\mathcal{A}(x)$ denotes the set of all self-financing trading strategies starting from initial endowment $(\varphi_0^0, \varphi_0^1) = (x, 0)$.

If we define

$$g = \varphi_T^0 + \varphi_T^1[(1-\lambda)S_T - c],$$

then we can rewrite（4）as

$$\max \mathbb{E}[U(g)], \quad g \in \mathcal{C}(x) \qquad (5)$$

where

$$\mathcal{C}(x) = \{\varphi_T^0 + \varphi_T^1[(1-\lambda)S_T - c] | \varphi \in \mathcal{A}(x)\} \subseteq L_+^0(P) \qquad (6)$$

denotes the set of terminal wealth at time T after the investor liquidates his position in risky assets to riskless assets.

So the primal problem for the investor is to maximize the expected utility of terminal wealth in the sense of semi-variable transaction costs

$$u(x) := \sup \mathbb{E}[U(g)] \quad g \in \mathcal{C}(x). \qquad (7)$$

2.2Existence and uniqueness of solutions for the primal problem

This subsection aims to prove the existence and uniqueness of solution for the problem（7）.

First of all, we give the definition and property of option strong super-martingale in the sense of semi-variable transaction costs:

Definition 2.2.1 An option process $X = (X_t)_{0\le t\le T}$ is called an option strong super-martingale, if for all stopping time $0 \le \sigma \le \tau \le T$ it satisfies

$$\mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] \le X_\sigma, \qquad (8)$$

in which we suppose that X_τ is integrable.

According to Doob-Meyer decomposition, an option process X is called an option strong super-martingale if and only if it can be decomposed into

$$X = M - A, \qquad (9)$$

where M is a local martingale as well as a super-martingale and A is an increasing predictable process.

Lemma 2.2.2 Assume that $\varphi = (\varphi^0, \varphi^1)$ satisfied self-financing trading strategies in the sense of semi-variable costs. Fix risky asset process S as above, suppose that there is a price process $\tilde{S}_t$ which satisfies $\tilde{S}_t \in [(1-\lambda)S_t - C, S_t]$, and there exists a probability measure Q such that $\tilde{S}_t$ is a martingale under Q. Then for all stopping time $0 \le \sigma \le \tau \le T$ the process

$$\tilde{V}(\varphi) := \varphi_t^0 + \varphi_t^1 \tilde{S}_t \tag{10}$$

is an option strong super-martingale under Q.

Proof: As what we have discussed, we should proof that $\tilde{V}(\varphi)$ can be decomposed as in (9).

According to the definition, we have

$$d\tilde{V}(\varphi) = \left(d\varphi_t^0 + \tilde{S}_t d\varphi_t^1\right) + \varphi_t^1 d\tilde{S}_t \tag{11}$$

so that

$$\tilde{V}(\varphi) = \int_0^t \left(d\varphi_u^0 + \tilde{S}_u d\varphi_u^1\right) + \int_0^t \varphi_u^1 d\tilde{S}_u \,. \tag{12}$$

We may use self-financial trading strategies

$$\begin{cases} d\varphi_t^{0,+} = [(1-\lambda)S_t - C]d\varphi_t^{1,-} \\ d\varphi_t^{0,-} = S_t d\varphi_t^{1,+} \end{cases}$$

for the first term of (12) so that there exists $d\varphi_t^0 + S_t d\varphi_t^1 = -(\lambda + C)d\varphi_t^{1,-}$, which is a decrease process. And for the second term of (12), it defines a martingale under Q measure as $\tilde{S}$ is so. Hence (12) is an optional strong super-martingale. □

In Definition 2.1.3, we assume that trading strategies $\varphi = (\varphi_t^0, \varphi_t^1)_{0\le t\le T}$ have finite variation. Our next lemma 2.2.3 proves that this assumption goes true.

Lemma 2.2.3 Assume that $\varphi = (\varphi^0, \varphi^1)$ satisfied self-financing trading strategies in the sense of semi-variable costs. Fix risky asset process S as above, suppose that there is a price process $\tilde{S}_t$ which satisfies $\tilde{S}_t \in [(1-\lambda)S_t - C, S_t]$, and there exists a probability measure Q such that $\tilde{S}_t$ is a martingale under Q. Then the total variation of φ remains bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$.

Proof: We can rewrite $\varphi^0 = \varphi^{0,+} - \varphi^{0,-}$ and $\varphi^1 = \varphi^{1,+} - \varphi^{1,-}$ as the sum of two increasing functions, respectively. Fix$\lambda' < \lambda < 1$, we define a new process φ^* by

$$\varphi^* = ((\varphi^0)^*, (\varphi^1)^*) = \left(\varphi_t^0 + \frac{(\lambda-\lambda')S_t}{(1-\lambda)S_t - C}\varphi_t^{0,+}, \varphi_t^{1,+}\right).$$

Obviously, φ^* is also a self-financing process under transaction costs$(\lambda' S_t + C)$, and $\frac{(\lambda-\lambda')S_t}{(1-\lambda)S_t - C}\varphi_t^{0,+}$ is the amount that the investor can get more under transaction costs$(\lambda' S_t + C)$ than in$(\lambda S_t + C)$.

By Lemma 2.2.2 we can find that

$$\left((\varphi^0)^*, (\varphi^1)^*\tilde{S}_t\right)_{0\le t\le T} = \left(\varphi_t^0 + \frac{(\lambda-\lambda')\tilde{S}_t}{(1-\lambda)\tilde{S}_t - C}\varphi_t^{0,+} + \varphi_t^{1,+}\tilde{S}_t\right)_{0\le t\le T}$$

is an optional strong super-martingale. Hence at terminal time T we have

$$\mathbb{E}_Q\left[\varphi_T^0 + \varphi_T^1\tilde{S}_T\right] + \mathbb{E}_Q\left[\frac{(\lambda-\lambda')\tilde{S}_T}{(1-\lambda)\tilde{S}_T - C}\varphi_T^{0,+}\right] \le x, \tag{13}$$

in which

$$\mathbb{E}_Q\left[\frac{(\lambda-\lambda')\tilde{S}_T}{(1-\lambda)\tilde{S}_T - C}\varphi_t^{0,+}\right] = \frac{(\lambda-\lambda')}{1-\lambda}\left[1 + C\mathbb{E}_Q\left[\frac{1}{(1-\lambda)\tilde{S}_T - C}\varphi_T^{0,+}\right]\right] \tag{14}$$

Because the investor liquidate his position in risky assets at terminal time T, then we have in (13) that $\varphi_T^1 = 0$. So (13) can be rewritten as

$$\mathbb{E}_Q\left[\frac{(\lambda-\lambda')\tilde{S}_T}{(1-\lambda)\tilde{S}_T - C}\varphi_T^{0,+}\right] \le x,$$

hence we have

$$\mathbb{E}_Q\left[\frac{1}{(1-\lambda)\tilde{S}_T - C}\varphi_T^{0,+}\right] \le \frac{x - \frac{\lambda-\lambda'}{1-\lambda}}{\frac{(\lambda-\lambda')C}{1-\lambda}} = \frac{(1-\lambda)x - (\lambda-\lambda')}{(\lambda-\lambda')C} = A. \tag{15}$$

On the other hand, (15) can be rewritten as

$$\mathbb{E}_Q\left[\frac{1}{(1-\lambda)\tilde{S}_T - C}\varphi_T^{0,+}\right] = \mathbb{E}_Q\left[\frac{1}{(1-\lambda)\tilde{S}_T - C}\right]\mathbb{E}_Q\left[\varphi_T^{0,+}\right] + \rho\sigma_1\sigma_2, \tag{16}$$

where$\sigma_1$and σ_2 denote the variance of$\frac{1}{(1-\lambda)\tilde{S}_T-C}$ and $\varphi_T^{0,+}$, respectively. If we denote $\mathbb{E}_Q[\tilde{S}_T]=\mu$ and $\mathrm{Var}[\tilde{S}_T]=\sigma^2$, then by Taylor series expansion at $\mathbb{E}_Q[\tilde{S}_T]$, we have

$$\mathbb{E}_Q\left[\frac{1}{(1-\lambda)\tilde{S}_T-C}\right]\approx\frac{1}{(1-\lambda)\mathbb{E}_Q[\tilde{S}_T]-C}+\frac{2(1-\lambda)^2}{[(1-\lambda)\mathbb{E}_Q[\tilde{S}_T]-C]^3}\times\frac{\sigma^2}{2}.$$

For（15）and the properties ofρwe have

$$-1\le\rho\le\frac{A-\mathbb{E}_Q\left[\frac{1}{(1-\lambda)\tilde{S}_T-C}\right]\mathbb{E}_Q[\varphi_T^{0,+}]}{\sigma_1\sigma_2},$$

whereρ denotes the correlation coefficient. Therefore

$$\mathbb{E}_Q[\varphi_T^{0,+}]\le\frac{A+\sigma_1\sigma_2}{\mathbb{E}_Q\left[\frac{1}{(1-\lambda)\tilde{S}_T-C}\right]}.$$

Note that $\varphi_T^0=\varphi_T^{0,+}-\varphi_T^{0,-}\ge 0$, we just have $\mathbb{E}_Q[\varphi_T^{0,-}]\le\mathbb{E}_Q[\varphi_T^{0,+}]$. Then

$$\mathbb{E}_Q[\varphi_T^{0,+}+\varphi_T^{0,-}]\le 2\frac{A+\sigma_1\sigma_2}{\mathbb{E}_Q\left[\frac{1}{(1-\lambda)\tilde{S}_T-C}\right]}=K. \qquad (17)$$

Finally if we use Chebyshev's inequality, we can easily get that

$$\mathbb{P}[\varphi_T^{0,+}+\varphi_T^{0,-}\ge\delta K]<\varepsilon.$$

As for the total variation of φ_T^1, from the self-financial strategies we have

$$d\varphi_t^{1,+}=\frac{d\varphi_t^{0,-}}{S_t}, \qquad (18)$$

By the assumption that S_t is strictly positive, we can control $\varphi_T^{1,+}$ by（18）and estimate $\varphi_T^{0,-}$ by（17）. Finally we can control $\varphi_T^{1,-}$ just using $\varphi_T^{1,+}-\varphi_T^{1,-}=\varphi_T^1-\varphi_0^1$. □

Our next lemma proves the set of trading strategies is closed.

Lemma 2.2.4 Under Assumption 2.1.1, the set $\mathcal{C}(x)$ is closed in L_+^0.

Proof: As what has been showed in Lemma2.2.3and the fact that investor cannot short any position at any time in this financial market, $\varphi_T^0$and φ_T^1 are bounded in L_+^0.

Let $\{\varphi_T^{0,n}\}_{n=1}^{\infty}$ be a nonnegative sequence in $\mathcal{C}(x)$ converging to some nonnegative $\varphi_T^0\in L_+^0$. We have to show $\varphi_T^0\in\mathcal{C}(x)$. We can find self-financing strategies$\varphi^n=(\varphi_t^{0,n},\varphi_t^{0,n})_{0\le t\le T}$, which start at $\varphi_0^n=(x,0)$ and end with $\varphi_T^n=(\varphi_T^{0,n},0)$ at terminal time T. Then these processes can be decomposed into $\varphi_T^{0,n}=\varphi_T^{0,n,+}-\varphi_T^{0,n,-}$. Just as what we have showed in Lemma 2.2.3, $(\varphi_T^{0,n,+})_{n=1}^{\infty}$and $(\varphi_T^{0,n,-})_{n=1}^{\infty}$as well as their convex combinations are bounded in L_+^0. By Lemma A1.1a in [14] we can find convex combinations converging a.s. to elements $\varphi_T^{0,+}$and $\varphi_T^{0,-}$. Therefore we have

$$\varphi_T^0=\varphi_T^{0,+}-\varphi_T^{0,-}\in\mathcal{C}(x).$$

In addition, for each rational time $t\in[0,T)$, assume that $(\varphi_t^{0,n,+})_{n=1}^{\infty}$, $(\varphi_t^{0,n,-})_{n=1}^{\infty}$, $(\varphi_t^{1,n,+})_{n=1}^{\infty}$ and $(\varphi_t^{1,n,-})_{n=1}^{\infty}$converge to some elements $\tilde{\varphi}_r^{0,+}$, $\tilde{\varphi}_r^{0,-}$, $\tilde{\varphi}_r^{1,+}$ and $\tilde{\varphi}_r^{1,-}$. By passing to a diagonal subsequence, we can suppose that this convergence holds true for all rational time $t\in[0,T)$, and the four limit processes are increasing and bounded in L_+^0.

So if we define the process $(\varphi_t^0,\varphi_t^1)_{0\le t\le T}$as $(\varphi_t^{0,+}-\varphi_t^{0,-},\varphi_t^{1,+}-\varphi_t^{1,-})_{0\le t\le T}$, it is easy to see that this is predictable, nonnegative and satisfies the self-financing conditions because the processes $(\varphi^n)_{n=1}^{\infty}$ is convergence for all $t\in[0,T]$. □

Finally we give a proof of existence and uniqueness of solutions for the primal problem.

Theorem 2.2.5 Assume that the utility function satisfies Assumption 2.1.5, the price process of risky asset satisfies Assumption 2.1.1. Moreover, we assume

$$u(x)<\infty,$$

then the solution of utility maximization problem（7）$\hat{g}\in\mathcal{C}(x)$exists and uniqueness.

Proof: The uniqueness of the solution is given by contradiction.

If there are two solutions for the utility maximization problem, suppose they are $\hat{f}$ and $\hat{g}$ which belong to $\mathcal{C}(x)$. So for every $\alpha \in [0,1]$, $[\alpha\hat{f} + (1-\alpha)\hat{g}]$, which is the convex combination of $\hat{f}$ and $\hat{g}$, should belong to $\mathcal{C}(x)$. However, since utility function U is concave, we have

$$\mathbb{E}[U(\alpha\hat{f} + (1-\alpha)\hat{g})] > \alpha\mathbb{E}[U(\hat{f})] + (1-\alpha)\mathbb{E}[U(\hat{g})]$$
$$= \alpha\mathbb{E}[U(\hat{f})] + \mathbb{E}[U(\hat{g})] - \alpha\mathbb{E}[U(\hat{g})] = \mathbb{E}[U(\hat{g})],$$

which is contradiction to the optimality of $\mathbb{E}[U(\hat{g})]$.

Next we prove the existence of the solution to the utility maximization problem.

First of all, since $u(x) < \infty$, we can find a maximizing sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}(x)$, in other words,

$$u(x) = \lim_{n\to\infty} \mathbb{E}[U(g_n)].$$

If we pass to a sequence of convex combinations $g_n := \operatorname{conv}(g_n, g_{n+1}, g_{n+2} \dots\dots)$, by Lemma A1.1a in [14] and Lemma 2.2.4, we can suppose that g_n converges to $\hat{g} \in \mathcal{C}(x)$.

So we can also prove that $\hat{g}$ is the solution to（7）by contradiction. If not, there exists a $\delta \in (0,\infty]$ such that

$$\delta = u(x) - \mathbb{E}[U(\hat{g})] > 0.$$

Assume that A_n and A_m are two disjoint sets. By Lemma 3.16 in [15], there exists

（1）$U(g_n) \geq \varepsilon^{-1}$ in A_n and $U(g_m) \geq \varepsilon^{-1}$ in A_m;

（2）$\mathbb{E}[U(g_n)\mathbb{I}_{A_n}] > \delta - \varepsilon$ and $\mathbb{E}[U(g_m)\mathbb{I}_{A_m}] > \delta - \varepsilon$;

（3）$\mathbb{E}[U(g_n)\mathbb{I}_{\Omega\setminus(A_n\cup A_m)}] > E[U(\hat{g})] - \varepsilon$ and $\mathbb{E}[U(g_m)\mathbb{I}_{\Omega\setminus(A_n\cup A_m)}] > E[U(\hat{g})] - \varepsilon$.

Then we have

$$\mathbb{E}\left[U\left(\frac{g_n + g_m}{2}\right)\right] = \mathbb{E}\left[U\left(\frac{g_n + g_m}{2}\right)\mathbb{I}_{(A_n\cup A_m)}\right] + \mathbb{E}\left[U\left(\frac{g_n + g_m}{2}\right)\mathbb{I}_{\Omega\setminus(A_n\cup A_m)}\right].$$

On the other hand, from Assumption 2.1.6 there exists $\gamma > 1$ that $U\left(\frac{x}{2}\right) \geq \frac{\gamma}{2}U(x)$. Then

$$\mathbb{E}\left[U\left(\frac{g_n + g_m}{2}\right)\mathbb{I}_{(A_n\cup A_m)}\right] \geq \frac{\gamma}{2}\mathbb{E}[U(g_n + g_m)\mathbb{I}_{(A_n\cup A_m)}] \geq \gamma(\delta - \varepsilon). \qquad (19)$$

In addition, since U is concave, then

$$\mathbb{E}\left[U\left(\frac{g_n + g_m}{2}\right)\mathbb{I}_{\Omega\setminus(A_n\cup A_m)}\right] \geq \frac{1}{2}\{\mathbb{E}[U(g_n)\mathbb{I}_{\Omega\setminus(A_n\cup A_m)}] + \mathbb{E}[U(g_m)\mathbb{I}_{\Omega\setminus(A_n\cup A_m)}]\}$$
$$\geq \mathbb{E}[U(\hat{g})] - \varepsilon. \qquad (20)$$

From（19）and（20）we have

$$\mathbb{E}\left[U\left(\frac{g_n + g_m}{2}\right)\right] \geq \mathbb{E}[U(\hat{g})] + \delta + [\delta(\gamma - 1) - \varepsilon(\gamma - 1)].$$

Since ε can be arbitrarily small, we can assume $[\delta(\gamma - 1) - \varepsilon(\gamma - 1)]$ is positive. But g_n is a maximizing sequence and $u(x) = \mathbb{E}[U(\hat{g})] + \delta$ is supremum, so it appears a contradiction, which means we have proved $\hat{g}$ is the solution to（7）. □

III. Utility maximization problem in shadow market

In this section, we first introduce the notion and then establish the shadow market which can produce the same maximum utility with the market in semi-variable cost. From the property of the shadow market and the friction market, we finally give the analytical formula for the original problem with semi-variable cost.

3.1 Modeling of shadow problem

In the market with transaction cost, consistent price system plays an important role. In order to establish the shadow problem and to solve the utility maximization problem with semi-variable cost, we first give the definition of consistent price systems as follows:

Definition 3.1.1 Assume that the price process of risky asset satisfies Assumption 2.1.1. Under semi-variable cost, a pair（$\tilde{S}, Q$）is called a consistent price system if it satisfies

（1）$\tilde{S}$ takes value in the bid and ask spread $[(1-\lambda)S_t - C, S_t]$;

（2）$\tilde{S}$ is a martingale under Q measure.

In addition, $\tilde{S}$ can be also written as the ratio of two super-martingales. In other words, for all $t \in [0,T]$,

$$\tilde{S} := \frac{Z_t^1}{Z_t^0} \in [(1-\lambda)S_t - C, S_t], \qquad a.s., \tag{21}$$

and the set of all consistent price systems is denoted by Z_λ.

Assumption 3.1.2 Under transaction costs $(\lambda' S_t + C)$ in trading one risky asset, for some $0 < \lambda' < \lambda$, we have $Z_{\lambda'} \neq \emptyset$.

Next we consider constructing a market with one zero interest rate bond and an risky asset whose price process is $\tilde{S}$:

Definition 3.1.3 An R^2-valued, predictable, finite variation process$\tilde{\varphi} = (\tilde{\varphi}_t^0, \tilde{\varphi}_t^1)_{0 \le t \le T}$ is called a self-financing trading strategy in frictionless market, if for all $0 \le t \le T$, $\tilde{\varphi}_t^1$ is integrable under $\tilde{S}$ and

$$\tilde{\varphi}_t^0 + \tilde{\varphi}_t^1 \tilde{S}_t = x + \int_0^t \tilde{\varphi}_u^1 \, d\tilde{S}_u.$$

We also assume in frictionless market that no assets can be shorted, so the shadow problem is established by all acceptable strategies defined as follows:

Definition 3.1.4 Aprocess$\tilde{\varphi}$is called acceptable,if for all $0 \le t \le T$, we have

$$\tilde{\varphi}_t^0 \ge 0 \text{ and } \tilde{\varphi}_t^1 \ge 0.$$

Therefore, the shadow problem can be written as

$$\max \mathbb{E}[U(\tilde{g})], \tilde{g} \in \tilde{\mathcal{C}}(x) \tag{22}$$

where

$$\tilde{\mathcal{C}}(x) = \{\tilde{g} \,|\, \tilde{\varphi} = (\tilde{\varphi}_T^0, \tilde{\varphi}_T^1) \in \tilde{\mathcal{A}}(x)\} \subseteq L_+^0(P)$$

denotes the terminal wealth for the investor in frictionless market at time T, and $\tilde{g} = \tilde{\varphi}_T^0 + \tilde{\varphi}_T^1 \tilde{S}_T$. In addition, $\tilde{\mathcal{A}}(x)$ denotes the set of all acceptable strategies starting from initial wealth $(\tilde{\varphi}_0^0, \tilde{\varphi}_0^1) = (x, 0)$.

Lemma 3.1.5 Assume$Z_{\lambda'} \neq \emptyset$. Then for all $0 \le t \le T$ we have

$$\tilde{\varphi}_t^0 + \tilde{\varphi}_t^1 \tilde{S}_t \ge \varphi_t^0 + \varphi_t^1 \tilde{S}_t, \tag{23}$$

Proof: For every $(\varphi^0, \varphi^1) \in \mathcal{A}(x)$, using the integration by parts formula we have

$$\varphi_t^0 + \varphi_t^1 \tilde{S}_t = x + \int_0^t d\varphi_u^0 + \int_0^t \varphi_u^1 d\tilde{S}_u + \int_0^t \tilde{S}_u d\varphi_u^1 \le x + \int_0^t \varphi_u^1 d\tilde{S}_u,$$

if we define

$$\begin{cases} \tilde{\varphi}_t^0 := \varphi_0^0 + \int_0^t \varphi_u^1 d\tilde{S}_u - \varphi_t^1 \tilde{S}_t, \\ \tilde{\varphi}_t^1 := \varphi_t^1, \end{cases} \tag{24}$$

then

$$\tilde{\varphi}_t^0 + \tilde{\varphi}_t^1 \tilde{S}_t = x + \int_0^t \varphi_u^1 d\tilde{S}_u \ge \varphi_t^0 + \varphi_t^1 \tilde{S}_t,$$

which means Lemma 3.1.5 has been proved. □

Note that for every $Z \in Z_\lambda$, we have $\mathcal{C}(x) \subseteq \tilde{\mathcal{C}}(x)$. Thus if we define

$$\tilde{u}(x) := \sup \mathbb{E}\,[U(\tilde{g})] \tilde{g} \in \tilde{\mathcal{C}}(x), \tag{25}$$

and compare with Lemma 3.1.5, then

$$u(x) \le \tilde{u}(x), \tag{26}$$

which means that transactions in frictionless market are always better than that in friction market. Hence we consider a proper price process $\hat{S}$ which can let inequality in（26）becomes equality. If so, the process is called shadow price.

Definition 3.1.6 Assume that consistent price system exists. Fixed the initial wealth x, the price process $\hat{S} := \frac{\hat{Z}^1}{\hat{Z}^0}$ is called shadow price, if there exists

$$\sup_{g \in \mathcal{C}(x)} \mathbb{E}[U(g)] = \sup_{\tilde{g} \in \tilde{\mathcal{C}}(x)} \mathbb{E}[U(\tilde{g})], \tag{27}$$

Next we shall first prove that shadow price exists and then construct it under semi-variable transaction costs.

Let V denotes the convex conjugate function of U defined by

$$V(y) := \sup_{x>0}\{U(x) - xy\}, y > 0,$$

then V is strictly decreasing, convex and continuously differentiable and satisfies

$$V'(0) = -\infty,\ V'(\infty) = 0,\ V(0) = \infty,\ V(\infty) = 0。$$

By definition,

$$V(Z_T^0) := \sup_{x>0}\{U(\tilde{g}) - Z_T^0\tilde{g}\}.$$

For every $\tilde{g} \in \tilde{\mathcal{C}}(x)$ under consistent price system,

$$\mathbb{E}[U(\tilde{g})] \leq \mathbb{E}[V(Z_T^0)] + \mathbb{E}[(Z_T^0\tilde{g})]. \tag{28}$$

By Lemma 4.1 in [7], $Z\varphi$ is a super-martingale process, thus we have

$$\mathbb{E}[\hat{Z}_T^0\hat{g}] \leq \hat{Z}_0^0 x, \tag{29}$$

So if there exists shadow price, then Lemma 4.1.10 should exist first:

Lemma 3.1.7 Let Assumption 2.1.1, 2.1.5, 2.1.6 and 3.1.2 hold. Then there exists $\hat{Z} \in Z_\lambda$ such that

（1）$\hat{Z}_T^0 = U'(\hat{g})$;

（2）$\mathbb{E}[\hat{Z}_T^0\hat{g}] = \hat{Z}_0^0 x$.

Proof: For each $0 < \alpha < 1$, we have $u(\alpha x) \geq \mathbb{E}[U(\alpha\hat{g})]$. Note that u is concave, so

$$\hat{Z}_0^0(x - \alpha x) \leq u(x) - u(\alpha x) \leq \mathbb{E}[U(\hat{g})] - \mathbb{E}[U(\alpha\hat{g})].$$

Then by the property of utility function U, we have

$$\hat{Z}_0^0 x \leq \mathbb{E}[U'(\hat{g})\hat{g}] \leq \mathbb{E}[\hat{Z}_T^0\hat{g}]. \tag{30}$$

Compared（29）with（30）we prove Lemma 3.1.7. □

Theorem 3.1.8 The consistent price system which satisfies Lemma 3.1.7 defines the shadow price $\hat{S}$.

Proof: By Lemma 3.1.5 and Lemma 3.1.7 we have

$$\tilde{u}(x) \geq u(x) = \mathbb{E}[U(\hat{g})] = \mathbb{E}[V(\hat{Z}_T^0) + \hat{Z}_T^0\hat{g}] = \mathbb{E}[V(\hat{Z}_T^0)] + \hat{Z}_0^0 x. \tag{31}$$

By（29）we also have

$$\mathbb{E}[V(\hat{Z}_T^0)] + \hat{Z}_0^0 x \geq \mathbb{E}[U(\tilde{g})] = \tilde{u}(x). \tag{32}$$

Compared（31）with（32）we complete the proof of Theorem 3.1.8. □

Remark 3.1.9 As has been proved above, if shadow price indeed exists, then the optimal strategy $\tilde{\varphi}$ for the utility maximization problem in frictionless market is also the optimal strategy for the problem in friction market. Hence shadow price $\hat{S}$ is the least favorable price in frictionless market. Thus the optimal strategy $\hat{\varphi}$ in friction market only trades when $\hat{S}$ is at bid or ask price, in other words,

$$\{d\hat{\varphi}_t^1 > 0\} \subseteq \{\hat{S}_t = S_t\} \text{and} \{d\hat{\varphi}_t^1 < 0\} \subseteq \{\hat{S}_t = (1-\lambda)S_t - C\}.$$

3.2The expression of shadow price

This subsection gives the expression of shadow price. First we give the definition of $\tilde{Z} = (\tilde{Z}_t^0, \tilde{Z}_t^1)_{0\leq t\leq T}$, then because the shadow price should satisfy the consistent price system, we finally prove this conclusion.

Firstly, we give the following dynamic programming principle similar to Section 7 in [10].

Definition 3.2.1 For every self-financing trading strategy $\varphi = (\varphi^0, \varphi^1)$, define its value function as

$$J(\varphi_t) := \operatorname*{ess\,sup}_{g\in\mathcal{A}_t} \mathbb{E}[U(g) \mid \mathcal{F}_t], \tag{33}$$

where$\mathcal{A}_t$ is the set of all self-financing trading strategies under semi-variable cost and no short selling constraint for each asset in $t \in [0, t]$.

Lemma 3.2.2 For every optimal strategy $\hat{\varphi}$in $0 \leq s \leq t \leq T$, the value function is a martingale, i.e.

$$J(\hat{\varphi}_s) = \mathbb{E}[J(\hat{\varphi}_t) \mid \mathcal{F}_s].$$

Proposition 3.2.3 Define

$$\begin{cases} \tilde{Z}_t^0 := \lim\limits_{\epsilon\to 0}\dfrac{J(\hat{\varphi}_t^0 + \epsilon, \hat{\varphi}_t^1) - J(\hat{\varphi}_t^0, \hat{\varphi}_t^1)}{\epsilon}, \\ \tilde{Z}_t^1 := \lim\limits_{\epsilon\to 0}\dfrac{J(\hat{\varphi}_t^0, \hat{\varphi}_t^1 + \epsilon) - J(\hat{\varphi}_t^0, \hat{\varphi}_t^1)}{\epsilon}, \end{cases} \tag{34}$$

on $0 \leq t < T$, and

$$\begin{cases}\tilde{Z}_T^0 := U'(\widehat{\varphi}_T),\\ \tilde{Z}_T^1 := U'(\widehat{\varphi}_T)[(1-\lambda)S_T - C].\end{cases} \tag{35}$$

at terminal time T.

Then for all $0 \le t \le T$, $\tilde{Z}$ is Right Continuous with Left Limits and is a martingale. In addition, $\tilde{Z}$ satisfies

$$[(1-\lambda)S_t - C] \le \frac{\tilde{Z}_t^1}{\tilde{Z}_t^0} \le S_t, \text{a.s.} \tag{36}$$

which means $\tilde{Z}_t^1/\tilde{Z}_t^0$ is the shadow price under semi-variable cost.

Proof: Firstly, assume that $\varepsilon_1 > \varepsilon_2 > 0$. By Definition 3.2.1 and the property of U as a concave function, we have

$$J\left(\left[\frac{\varepsilon_2}{\varepsilon_1}(\widehat{\varphi}_t^0 + \varepsilon_1) + \left(1 - \frac{\varepsilon_2}{\varepsilon_1}\right)\right]\widehat{\varphi}_t^0, \widehat{\varphi}_t^1\right) \ge \frac{\varepsilon_2}{\varepsilon_1} J\big((\widehat{\varphi}_t^0 + \varepsilon_1), \widehat{\varphi}_t^1\big) + \left(1 - \frac{\varepsilon_2}{\varepsilon_1}\right) J(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1),$$

which can be also written as

$$\frac{J(\widehat{\varphi}_t^0 + \varepsilon_2, \widehat{\varphi}_t^1) - J(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)}{\varepsilon_2} \ge \frac{J(\widehat{\varphi}_t^0 + \varepsilon_1, \widehat{\varphi}_t^1) - J(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)}{\varepsilon_1}.$$

Since $\tilde{Z}_t^0$ is the limit of an increasing sequence, it is well-defined, and so on with $\tilde{Z}_t^1$.

Next, since the set of trading strategies $\mathcal{A}_t$ in（33）is directed upwards, then（33）can be rewritten by

$$J(\widehat{\varphi}_t + \varepsilon e_i) := \operatorname*{ess\,sup}_{g \in \mathcal{A}_t^{\widehat{\varphi}}} \mathbb{E}\big[U(g) \mid \mathcal{F}_t\big] = \lim_{n\to\infty} \mathbb{E}\big[U(g^n) \mid \mathcal{F}_t\big],$$

where $(g^n)_{n\ge 0}$ is the increasing sequences in $\mathcal{A}_t^{\widehat{\varphi}}$, and for simplicity

$$\tilde{Z}_t^i := \lim_{\varepsilon\to 0} \frac{J(\widehat{\varphi}_t + \varepsilon e_i) - J(\widehat{\varphi}_t)}{\varepsilon}. \quad i = 0,1, t \in [0,T]$$

Then, we may prove that $\tilde{Z}$ is super-martingale. Clearly $\mathcal{A}_t^{\widehat{\varphi}} \subseteq \mathcal{A}_s^{\widehat{\varphi}}$ for all $0 \le s \le t \le T$, so

$$J(\widehat{\varphi}_s + \varepsilon e_i) = \operatorname*{ess\,sup}_{g \in \mathcal{A}_s^{\widehat{\varphi}}} \mathbb{E}\big[U(g) \mid \mathcal{F}_s\big] \ge \operatorname*{ess\,sup}_{g \in \mathcal{A}_t^{\widehat{\varphi}}} \mathbb{E}\big[U(g) \mid \mathcal{F}_t\big] \ge \mathbb{E}\big[U(g^n) \mid \mathcal{F}_s\big].$$

By the monotone convergence theorem,

$$J(\widehat{\varphi}_s + \varepsilon e_i) \ge \lim_{n\to\infty} \mathbb{E}\big[\mathbb{E}\big[U(g^n) \mid \mathcal{F}_t\big] \mid \mathcal{F}_s\big] = \mathbb{E}\big[J(\widehat{\varphi}_t + \varepsilon e_i) \mid \mathcal{F}_s\big].$$

Hence by definition of $\tilde{Z}_t^i (i = 0,1)$, for $0 \le t < T$ we have

$$\tilde{Z}_s^i \ge \lim_{\varepsilon\to 0} \mathbb{E}\left[\frac{J(\widehat{\varphi}_t + \varepsilon e_i) - (\widehat{\varphi}_t)}{\varepsilon} \mid \mathcal{F}_s\right] = \mathbb{E}\left[\lim_{\varepsilon\to 0} \frac{J(\widehat{\varphi}_t + \varepsilon e_i) - (\widehat{\varphi}_t)}{\varepsilon} \mid \mathcal{F}_s\right] = \mathbb{E}\big[\tilde{Z}_t^i \mid \mathcal{F}_s\big]$$

And we still have to verify that $\tilde{Z}$ is also super-martingale when $t = T$.

By the monotone convergence theorem again, we have

$$\begin{aligned}\tilde{Z}_s^0 &:= \lim_{\varepsilon\to 0} \frac{J(\widehat{\varphi}_s^0 + \varepsilon, \widehat{\varphi}_s^1) - J(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1)}{\varepsilon} \ge \lim_{\varepsilon\to 0} \mathbb{E}\left[\frac{U(\widehat{\varphi}_T^0 + \varepsilon) - U(\widehat{\varphi}_T^0)}{\varepsilon} \mid \mathcal{F}_s\right]\\ &= \mathbb{E}\big[U'(\widehat{\varphi}_T^0) \mid \mathcal{F}_s\big] = \mathbb{E}\big[\tilde{Z}_T^0 \mid \mathcal{F}_s\big].\end{aligned}$$

$$\begin{aligned}\tilde{Z}_s^1 &:= \lim_{\varepsilon\to 0} \frac{J(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1 + \varepsilon) - J(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1)}{\varepsilon}\\ &\ge \lim_{\varepsilon\to 0} \mathbb{E}\left[\frac{U(\widehat{\varphi}_T^0 + \varepsilon[(1-\lambda)S_T - C]) - U(\widehat{\varphi}_T^0)}{\varepsilon} \mid \mathcal{F}_s\right]\\ &= \mathbb{E}\big[U'(\widehat{\varphi}_T^0)[(1-\lambda)S_T - C] \mid \mathcal{F}_s\big] = \mathbb{E}\big[\tilde{Z}_T^1 \mid \mathcal{F}_s\big].\end{aligned}$$

As for $\tilde{Z}_0^0$, because of $u(x) < \infty$, we have

$$\tilde{Z}_0^0 := \lim_{\varepsilon\to 0} \frac{J(\widehat{\varphi}_0^0 + \varepsilon, \widehat{\varphi}_0^1) - J(\widehat{\varphi}_0^0, \widehat{\varphi}_0^1)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{u(x+\varepsilon) - u(x)}{\varepsilon} < \infty.$$

Hence $\tilde{Z}$ is super-martingale.

Finally, we show that the ratio of $\tilde{Z}_t^1/\tilde{Z}_t^0$ is bounded between bid and ask price.

By definition of $\tilde{Z}$, this conclusion is obviously at terminal time T.

For all $t \in [0,T)$, let $(p_m^n)_{m\ge 0}$ be a partition of $[0,\infty)$. For each $\varepsilon > 0$, on every set $A_m := p_m^n < S_t \le p_{m+1}^n$ we have

$$(\widehat{\varphi}_t^0 + \varepsilon, \widehat{\varphi}_t^1) \ge J\left(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1 + \frac{\varepsilon}{S_t}\right) \ge J\left(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1 + \frac{\varepsilon}{p_{m+1}^n}\right).$$

Therefore

$$\frac{J(\widehat{\varphi}_t^0+\varepsilon,\widehat{\varphi}_t^1)-J(\widehat{\varphi}_t^0,\widehat{\varphi}_t^1)}{\varepsilon} \geq \frac{J\left(\widehat{\varphi}_t^0,\widehat{\varphi}_t^1+\frac{\varepsilon}{p_{m+1}^n}\right)-J(\widehat{\varphi}_t^0,\widehat{\varphi}_t^1)}{\frac{\varepsilon}{p_{m+1}^n}\times p_{m+1}^n}.$$

By the monotone convergence theorem,

$$\tilde{Z}_t^0 \mathbb{I}_{A_m} \geq \frac{1}{p_{m+1}^n}\tilde{Z}_t^1 \mathbb{I}_{A_m},$$

and thus

$$\sum_{m\geq 0}\mathbb{I}_{A_m}\tilde{Z}_t^0 \geq \sum_{m\geq 0}\mathbb{I}_{A_m}\frac{1}{p_{m+1}^n}\tilde{Z}_t^1.$$

When $n\to\infty$ we have

$$\frac{\tilde{Z}_t^1}{\tilde{Z}_t^0}\leq S_t,\quad 0\leq t\leq T.$$

On the other hand, on every set $A_m := p_m^n < S_t \leq p_{m+1}^n$ again we have

$$J(\widehat{\varphi}_t^0,\widehat{\varphi}_t^1+\varepsilon)\geq J(\widehat{\varphi}_t^0+\varepsilon[(1-\lambda)S_t-C],\widehat{\varphi}_t^1).$$

Following the same steps we finally get

$$\frac{\tilde{Z}_t^1}{\tilde{Z}_t^0}\geq (1-\lambda)S_t - C,\quad 0\leq t\leq T.$$

Last but not least, if we define

$$\begin{cases}\hat{Z}_t^i := \lim_{s\to t}\tilde{Z}_s^i, & 0\leq t\leq T;\\ \hat{Z}_t^i := \tilde{Z}_T^i, & t=T,\end{cases} \tag{37}$$

then by Proposition 1.3.14(i) in [17] we know that $\hat{Z}_t^1/\hat{Z}_t^0$ is a super-martingale and Right Continuous with Left Limits process, which is bounded between bid and ask price. Hence it defines the shadow price under semi-variable transaction cost. □

3.3The optimal strategy for the friction problem

Assume that in frictionless market, the price process of risky asset satisfies Geometric Brownian motion, i.e.

$$d\tilde{S}_t = r\tilde{S}_t dt + \sigma\tilde{S}_t dW_t,$$

wherer denotes the instantaneous expected rate of return of risky asset, σ denotes the instantaneous volatility, and W_t denotes the standard Brownian motion.

Let π_t denotes the proportion of risky asset in total assets at any time t, i.e.

$$\pi_t = \frac{\tilde{\varphi}_t^1\tilde{S}_t}{\tilde{\varphi}_t^0+\tilde{\varphi}_t^1\tilde{S}_t}, \tag{38}$$

where $\tilde{S}_t$ denotes the price of risky asset in frictionless market. Then by Lemma 3.1 in [18] the optimal proportion of risk asset in total assets should be

$$\pi_t = \frac{r}{\sigma^2}. \tag{39}$$

By（38）and（39）we have

$$\tilde{\varphi}_t^1 = \frac{r}{(\sigma^2-r)\tilde{S}_t}\tilde{\varphi}_t^0. \tag{40}$$

Because the risky asset in frictionless market only trades when $\tilde{S}_t = (1-\lambda)S_t - C$, so we have

$$\tilde{\varphi}_t^1 = \frac{r}{(\sigma^2-r)[(1-\lambda)S_t-C]}\tilde{\varphi}_t^0. \tag{41}$$

Finally by（24）we get

$$\widehat{\varphi}_t^1 = \frac{r}{\sigma^2[(1-\lambda)S_t-C]}\left(x+(1-\lambda)\int_0^t\widehat{\varphi}_u^1 dS_u\right), \tag{42}$$

which is the analytical formula for the original problem under semi-variable cost.

IV. Case analysis

This section gives an example in order to verify the rationality and effectiveness of our conclusion about shadow price under semi-variable cost.

Assume that there are only one riskless asset and one risky asset in the market. The riskless asset is zero interest rate and the price of risky asset is given as follows:

$$S_t = \begin{cases} exp\left(W_t + \frac{t}{2}\right), 0 \le t \le \frac{T}{2}; \\ S_{T/2}, \qquad \frac{T}{2} \le t \le T, \end{cases} \qquad (43)$$

where $(W_t)_{t\ge 0}$ is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$.

Fix proportional transaction cost rate $\lambda \in (0,1)$ and the constant transaction cost C, assume the initial wealth of the investor is $x = 1$, which belongs to the riskless asset at $t = 0$.

In addition, from the definition of shadow price, for every $t \in [0, T]$, we have

$$\tilde{S}_t = \begin{cases} S_0, & t = 0; \\ exp\left(W_t + \frac{t}{2} + \frac{2\ln(1-\lambda)t}{T}\right) - \frac{t}{T/2}C, & 0 < t < \frac{T}{2}; \\ (1-\lambda)S_{T/2} - C, & \frac{T}{2} \le t \le T. \end{cases} \qquad (44)$$

By（44）, we have optimal strategy under semi-variable cost

$$(\hat{\varphi}^0, \hat{\varphi}^1) = \begin{cases} (1,0), & t = 0; \\ (0,1), & 0 < t < \frac{T}{2}; \\ \left((1-\lambda)S_{T/2} - C, 0\right), & \frac{T}{2} \le t \le T. \end{cases} \qquad (45)$$

According to（45）, we calculate the utility of terminal wealth, and find that the utility in friction market and in frictionless market are the same, hence（45）is indeed the optimal strategy under semi-variable cost.

We can also know from（44）that with the increase of the price for risky assets with transaction costs, the shadow price will also increase; in addition, with the increase of the fixed cost C and the fixed proportional coefficientλ, the shadow price will decrease.

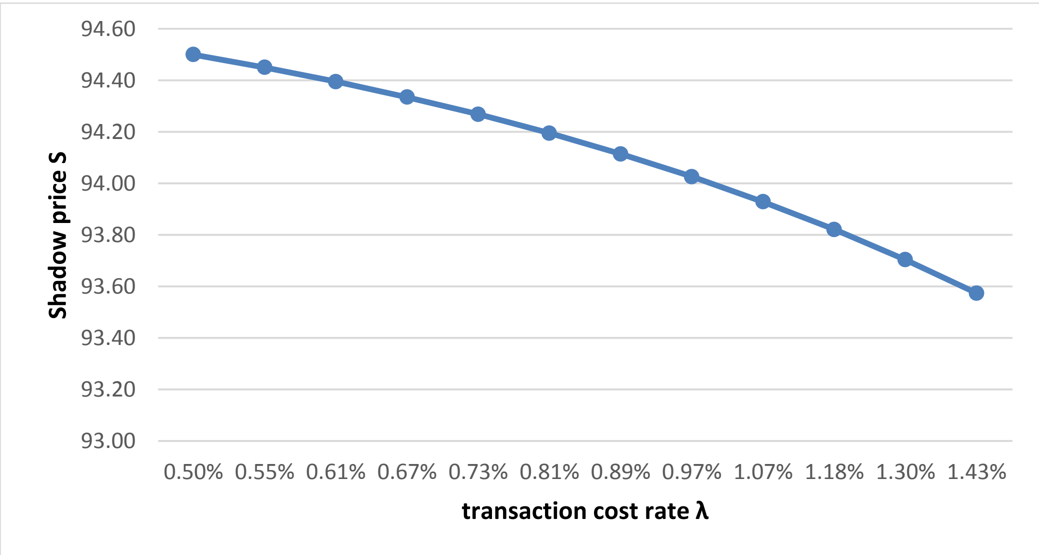

Figure 1: The relationship between shadow price $\tilde{S}$ and transaction cost rateλ

Figure 1 is the change of shadow price for every 10% increase of transactioncost rate. With the increase of transaction cost rateλ, shadow price$\tilde{S}$decreases gradually.

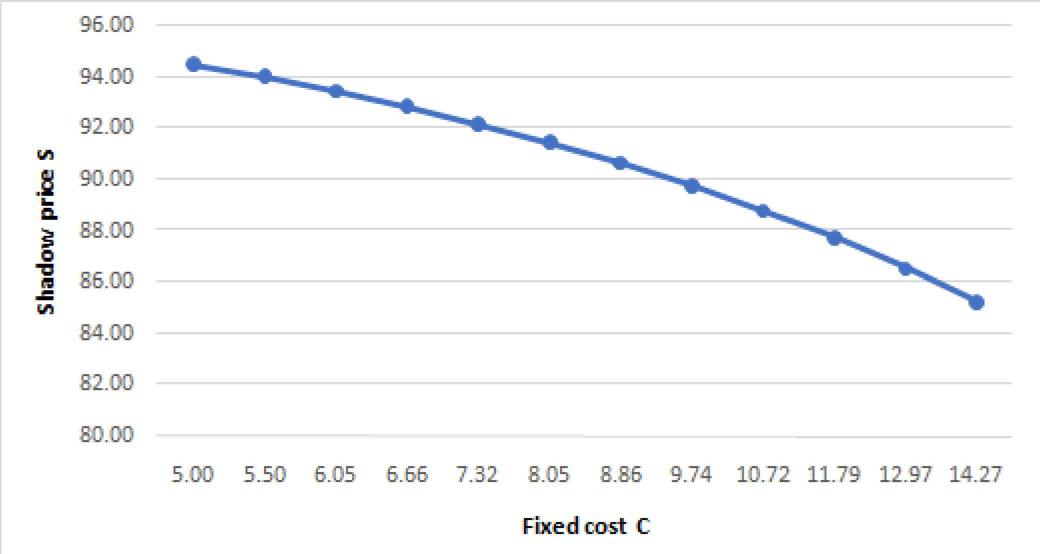

Figure 2: The relationship between shadow price $\tilde{S}$ and fixed cost C

Figure 2 is the change of shadow price for every 10% increase of fixedcost C. With the increase of fixed cost C, shadow price $\tilde{S}$ decreases gradually.

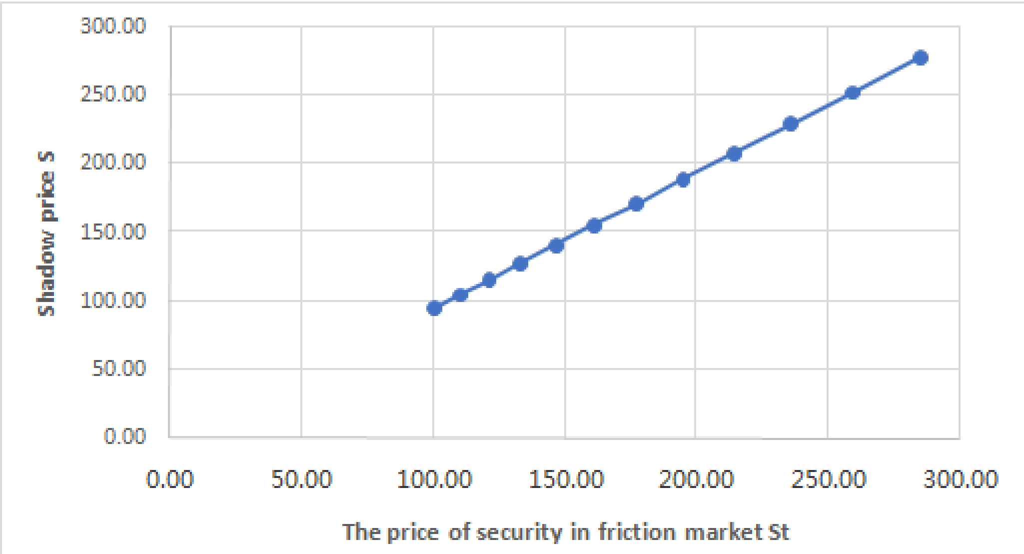

Figure 3: The relationship between shadow price $\tilde{S}$ and the price of security in friction market S_t

Figure 3 is the change of shadow price for every 10% increase of the priceof security in friction market S_t. With the increase of the price of securityin friction market St, shadow price ˜S also increases.

V. Conclusion

Based on real market transactions, we consider an asset allocation problem in the form of semi-variable transaction costs, and prove the existence of shadow price. Then we give the analytical formula for the problem with friction under the property of shadow price with the price of security in friction market. Our conclusion provides another way to solve the expected utility maximization problem with constraints.

REFERENCES

[1] Pliska.S.R. A stochastic calculus model of continuous trading: optimal portfolios.*Mathematics of Operations Research,11(2),*1986, 370-382.

[2] Karatzas I, Lehoczky J P, Shreve S E, et al. Martingale and duality methods for utility maximization in a incomplete market.*SIAM Journal on Control and Optimization,29(3),*1991, 702-730.

[3] Kramkov D, Schachermayer W. The asymptotic elasticity of utility functions and optimal investment in incomplete markets,*The Annals of Applied Probability,9(3),* 1999, 904-950.

[4] Liu H, Mark L. Optimal Portfolio Selection with Transaction Costs and Finite Horizons,*Review of Financial Studies, 15(3),* 2002, 805-835.

[5] Cvitanić J, KaratzasI . Hedging and portfolio optimization under transaction cost: a martingale approach, *Mathematical Finance, 6(2),* 1996, 133-165.

[6] JKallsen J, Muhle-Karbe J. Existence of shadow prices in finite probability spaces, *Mathematical Methods of Operations Research, 73(2),* 2011, 251-262

[7] Benedetti G, Campi L, Kallsen J, et al. On the existence of shadow prices, *Finance and Stochastics, 17(4),*2013, 801-818.

[8] Czichowsky C, Schachermayer W. Duality theory for portfolio optimization under transaction costs, *Annals of Applied Probability, 26(3),*2016, 1888-1941.

[9] Bayraktar E, Yu X. Optimal investment with random endowments and transaction costs: duality theory and shadow prices, *Mathematics and Financial Economics, 13(2),* 2018,253-286.

[10] Loewenstein M. On optimal portfolio trading strategies for an investor facing transactionscosts in a continuous trading market, *Journal of Mathematical Economics,33(2),*2000, 209-228.

[11] Kallsen J, Muhle-Karbe J. On using shadow prices in portfolio optimizationwith transaction costs, *Annals of Applied Probability, 20(4),*2010, 1341-1358.

[12] GerholdS, Muhle-Karbe J. and SchachermayerW. The dual optimizer forthe growth-optimal portfolio under transaction costs, *Finance and Stochastics,17(2),*2013, 325-354.

[13] HerczeghA,Prokaj V. Shadow price in the power utility case, *Annals of AppliedProbability, 25(5),* 2015, 2671-2707.

[14] Delbaen F, Schachermayer W. A general version of the fundamental theorem of asset pricing, *MathematischeAnnalen, 300(1),* 1994, 463-520.

[15] Schachermayer W.*Portfolio optimization in incomplete financial markets*(CattedraGalileiana. [Galileo Chair]. ScuolaNormaleSuperiore, Classe di Scienze, Pisa, 2004).

[16] Karoui N.E.,*Les aspectsprobabilistes du contrôlestochastique*(Springer: Berlin,1979).

[17] Karatzas I. ShreveS.E.Brownian motion and stochastic calculus, *Graduate Texts in Mathematics, 113, 30(1),* 1989, 262-263

[18] Goll T, Kallsen J.Optimal portfolios for logarithmic utility, *Stochastic Processes & Their Applications, 89(1),*2000, 31-48.