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Optimal investment and reinsurance for mean-variance insurers under variance premium principle

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Abstract: *This paper studies an optimal investment and reinsurance problem for a jump-diffusion risk model with short-selling constraint under the mean-variance criterion. Assume that the insurer is allowed to purchase proportional reinsurance from the reinsurer and invest in a risk-free asset and a risky asset whose price follows a geometric Brownian motion. In particular, both the insurance and reinsurance premium are assumed to be calculated via the variance principle with different parameters respectively. The diffusion term can explain the uncertainty associated with the surplus of the insurer (U-S case) or the additional small claims (A-C case), which are the uncertainty associated with the insurance market or the economic environment. By using techniques of stochastic control theory, with normal constraints on the control variables, closed-form expressions for the value functions and optimal investment-reinsurance strategies are derived in both A-C case and U-S case.*

Keywords: *Investment; Reinsurance; Variance premium; Mean-variance.*

I. Introduction

Recently, the topic about investment and reinsurance problem in insurance risk management has been extensively investigated in the literature. For example, Browne (1995) investigated the optimal investment strategies for an insurance company who maximized the expected utility of terminal wealth or minimizes the ruin probability with its surplus process modeled by a drifted Brownian motion. For the same diffusion model, Promislow and Young (2005) considered the problem of minimizing the probability of ruin subject to both investment and proportional reinsurance strategies. Aiming at maximizes expected utility, Irgens and Paulsen (2004) considered the optimal reinsurance and investment problem for an insurer whose surplus follows a classical risk process perturbed by a diffusion, where the compound Poisson process stands for the large claims and the diffusion term represents the additional small claims (A-C, for short). Yang and Zhang (2005) studied the optimal investment strategies for an insurer who maximized the expected exponential utility of the terminal wealth or maximize the survival probability, the surplus process is driven by a jump–diffusion model in which the diffusion term stands for the uncertainty associated with the surplus of the insurer (U-S, for short). Huang et al.(2016) considered the optimal control problem with constraints for an insurer with a jump–diffusion process under the expected value premium principle, they derived the closed form expressions of the optimal strategies and value function for A-C case and U-S case, respectively. We refer readers to Xu et al.(2008), Gu et al.(2010), Liang et al.(2011) and Guan and Liang (2014) for maximizing the expected utility of the insurers' terminal wealth in different situations.

Besides ruin probability minimization and expected exponential utility maximization, mean–variance

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criterion is another important objective function in finance, which is first proposed by Markowitz (1952), as a result, recently many scholars consider the optimal investment–reinsur-
ance policies for insurers under mean–variance criterion. For example, Bäuerle (2005) considered optimal proportional reinsurance/new business problem under mean–variance criterion with surplus process driven by classical Cramér Lundberg model. Bai and Zhang (2008) derived explicit optimal investment–reinsurance policies for an insurer under mean–variance criterion with constrained controls. Zeng et al.(2010) used stochastic maximum principle to discuss the optimal investment problem for an insurer with jump–diffusion risk process under benchmark and mean–variance criteria. For more detailed discussion, the readers are referred to Li et al.(2015), Zeng et al.(2013), Shen and Zeng (2014) for the continuous-time cases and reference therein.

As far as we are concerned, in most of the literature, the expected value premium principle is commonly used as the premium principle for mathematical convenience. Generally speaking, the expected value premium principle is commonly used in life insurance which has the stable and smooth claim frequency and claim sizes, while the variance premium principle is extensively used in property insurance. The variance premium principle permits the company to take the fluctuations of claims into consideration when pricing insurance contracts. Liang et al.(2011), Zhang et al.(2011), Zhou et al.(2015), Zhou et al.(2012) and Yao et al.(2014) discussed the problem of optimal proportional reinsurance and investment or solved the optimal dividend problem with proportional reinsurance under variance principle, but in whose papers only the reinsurance premium is calculated according to variance principle. Sun et al.(2017) considered a robust optimal investment and reinsurance problem under model ambiguity and default risk for an insurer, and aimed to maximize the minimal expected exponential utility, in whose paper both the insurance and reinsurance premium are assumed to be calculated via the variance principle.

Note that few literature uses the variance premium principle to calculate both insurance and reinsurance premium under the mean-variance criterion. Hence in this paper, we study the optimal reinsurance and investment problem for a jump–diffusion risk model with short-selling constraint and assume both the insurance and reinsurance premium payments are calculated by using the variance premium principle with different parameter respectively. To the best of our knowledge, the diffusion term in the jump-diffusion risk model is usually considered the uncertainty associated with the surplus of the insurer and/or the economic environment (U-S case), but the uncertainty may be related to the claims. Thus, in our paper, there are two kinds of different explanations about the diffusion term in the jump-diffusion risk model. One is the A-C case as described in Irgens and Paulsen (2004), the other is the U-S case is given as in Yang and Zhang (2005). Although Huang et al.(2016) and Zhou et al.(2015) studied the optimal investment and reinsurance problem for both A-C case and U-S case, they aimed to maximize the expected utility of terminal wealth which is different to our aim. We aim at investigating the optimal strategy under mean-variance criterion. The other difference is the premium principle. Both the insurance and reinsurance premium are calculated according to the expected value principle in Huang et al.(2016). Zhou et al.(2015) used the expected value principle to calculate the insurance premium, while the reinsurance premium is calculated via the variance premium principle. However, both the insurance and reinsurance premium are calculated by using the variance premium principle which is more suitable for our case.

The paper is organized as follows. The formulation of our model is presented in Section 2. Section 3 provides the mean-variance problem for an insurer. In Section 4, closed-form expressions for the value functions and optimal investment-reinsurance strategies are obtained for the U-S case, the results for the A-C case are given in Section 5. Comparing the results, we find the optimal reinsurance strategies and the optimal investment strategies are different for those two cases.

II. Model formulation

We start with a filtered complete probability space $(\Omega, F, \{F_t\}_{0\le t\le T}, P)$, where T is a finite and positive constant, representing the time horizon, F_t stands for the information of the market available up to time *t*. $[0,T]$ is a fixed time horizon. All the processes introduced below are assumed to be well-defined and adapted processes in this space. In addition, suppose that trading in the financial and insurance markets is continuous, without taxes or transaction costs, and that all assets are infinitely divisible.

Suppose that an insurer's surplus process follows a jump-diffusion risk model. In this model, without reinsurance and investment, the surplus X_t of the insurer at time t is

$$X_t = X_0 + ct - \sum_{i=1}^{N(t)} X_i + \sigma_0 W_t^{(0)} \tag{1}$$

where $X_0 \geq 0$ is the initial surplus, $c > 0$ is the premium rate, and $S_t = \sum_{i=1}^{N(t)} X_i$ is a compound Poisson process, representing the cumulative claims up to time t, where $\{N(t), t \geq 0\}$ is a homogeneous Poisson process with intensity λ, and $\{X_i, i \geq 1\}$ is a sequence of positive independent and identically distributed random variables with finite first moment $EX_i = \mu_1$ and second moment $EX_i^2 = \mu_2^2$. $\{W_t^{(0)}, t \geq 0\}$ is a standard Brownian motion, $\sigma_0 \geq 0$ is a constant. In addition, we assume that $\{X_i, i \geq 1\}$, $\{N(t), t \geq 0\}$ and $\{W_t^{(0)}, t \geq 0\}$ are mutually independent.

To the best of our knowledge, the diffusion term $\sigma_0 W_t^{(0)}$ in the model (1) usually has two kinds of different understanding in the literatures. On the one hand, $\sigma_0 W_t^{(0)}$ stands for the uncertainty associated with the surplus of the insurer at time t (that is, the U-S case), thus the aggregate claims up to time t is S_t. On the other hand, $\sigma_0 W_t^{(0)}$ represents the additional small claims, which are the uncertainty associated with the insurance market or the economic environment (that is, the A-C case)，let $\hat{S}_t$ be the corresponding aggregate claim process in this case, then we have

$$\hat{S}_t = S_t - \sigma_0 W_t^{(0)}.$$

The insurer can purchase proportional reinsurance from the reinsurer. For each $t \in [0, T]$, the proportional reinsurance level is denoted by the value of risk exposure $a_t \in [0,1]$. In this case, the cedent (insurer) pays $100(1-a_t)$% of the claim while the reinsurer pays the rest, i.e. 100 a_t % of the claim. When the claim X occurs, the insurer pays only $(1-a_t)X$ while the reinsurer pays $a_t X$. However, the insurer has to pay a premium at the rate of c^a to the reinsurer due to the reinsurance business. Throughout this article, we assume that both insurance premium and reinsurance premium are calculated according to the variance principle with different parameter respectively.

For the U-S case, the insurance premium under the variance principle takes the form

$$ct = E\left(\sum_{i=1}^{N(t)} X_i\right) + \theta_1 Var\left(\sum_{i=1}^{N(t)} X_i\right) = \lambda(\mu_1 + \theta_1 \mu_2^2)t,$$

where $Var(\cdot)$ stands for variance, and $\theta_1 > 0$ is the safety loading associated of the insurer with the variance of ceded risk. Meanwhile, the reinsurance premium under the variance principle is

$$c^a t = E\left(\sum_{i=1}^{N(t)} a_t X_i\right) + \theta_2 Var\left(\sum_{i=1}^{N(t)} a_t X_i\right) = \lambda[a_t \mu_1 + \theta_2 a_t^{\,2} \mu_2^2]t,$$

where $\theta_2 \in [\theta_1, +\infty)$ represents the safety loading of the reinsurer.

Thus, the insurer's surplus process in the presence of reinsurance can be written as

$$\mathrm{d}X_t^a = \lambda\left((1-a_t)\mu_1 + (\theta_1 - \theta_2 a_t^{\,2})\mu_2^2\right)\mathrm{d}t - \mathrm{d}\sum_{i=1}^{N(t)}(1-a_t)X_i + \sigma_0 \mathrm{d}W_t^{(0)},$$

The above process can be approximated by the following drifted process

$$\mathrm{d}X_t^a = \lambda\mu_2^2(\theta_1 - \theta_2 a_t^{\,2})\mathrm{d}t + (1-a_t)\sqrt{\lambda}\mu_2 \mathrm{d}W_t^{(1)} + \sigma_0 \mathrm{d}W_t^{(0)},$$

where $\{W_t^{(1)}, t \geq 0\}$ independent of $\{W_t^{(0)}, t \geq 0\}$ is a standard Brownian motion.

For the A-C case, the insurance premium under the variance principle is given by

$$ct = E\left(\sum_{i=1}^{N(t)} X_i + \sigma_0 W_t^{(0)}\right) + \theta_1 Var\left(\sum_{i=1}^{N(t)} X_i + \sigma_0 W_t^{(0)}\right) = \left(\lambda\mu_1 + \theta_1(\lambda\mu_2^2 + \sigma_0^2)\right)t.$$

and the aggregate reinsurance premium is

$$c^a t = E\left(\sum_{i=1}^{N(t)} a_t X_i + a_t\sigma_0 W_t^{(0)}\right) + \theta_2 Var\left(\sum_{i=1}^{N(t)} a_t X_i + a_t\sigma_0 W_t^{(0)}\right) = \left(a_t\lambda\mu_1 + \theta_2 a_t^2(\lambda\mu_2^2 + \sigma_0^2)\right)t.$$

With such a reinsurance strategy, the surplus process of the insurer follows

$$\mathrm{d}\hat{X}_t^a = \left(\lambda\mu_1(1-a_t) + (\lambda\mu_2^2+\sigma_0^2)(\theta_1 - \theta_2 a_t^2)\right)\mathrm{d}t - \mathrm{d}\sum_{i=1}^{N(t)}(1-a_t)X_i + (1-a_t)\sigma_0\mathrm{d}W_t^{(0)}.$$

The above process can be approximated by the following drifted process

$$\mathrm{d}\hat{X}_t^a = (\lambda\mu_2^2+\sigma_0^2)(\theta_1-\theta_2 a_t^2)\,\mathrm{d}t + (1-a_t)\sqrt{\lambda}\mu_2\mathrm{d}W_t^{(2)} + (1-a_t)\sigma_0\mathrm{d}W_t^{(0)},$$

where $\left\{W_t^{(2)}, t\ge 0\right\}$ is a standard Brownian motion, and it is independent of $\left\{W_t^{(0)}, t\ge 0\right\}$ and $\left\{W_t^{(1)}, t\ge 0\right\}$.

Moreover, we assume that the insurer is allowed to invest in a financial market consisting of a risk-free (e.g., a bound or a bank account) and a risky asset (e.g., a stock). The price process of the risk-free asset evolves according to

$$\mathrm{d}B_t = r_0 B_t \mathrm{d}t$$

where $r_0 > 0$, representing the risk-free interest rate, is a continuous bounded deterministic function. The price process of the risky asset is described by the GBM model

$$\mathrm{d}P_t = P_t\left(r_1\mathrm{d}t + \sigma\mathrm{d}W_t\right)$$

where $r_1(>r_0)$ and σ are positive constants that represent the expected instantaneous rate of the risky asset and the volatility of the risky asset price, respectively. $\{W_t, t\ge 0\}$ is another standard Brownian motion, independent of S_t and $\{W_t^{(i)}, t\ge 0\}, i=0,1,2.$

The insurer, starting from an initial capital x_0 at time 0, is allowed to dynamically purchase proportional reinsurance and invest in the financial market described above. A trading strategy is denoted by a pair of stochastic processes $\pi=(a_t, b_t)$, where a_t and b_t are the value of the risk exposure as described above and the total amount of money invested in the risky asset at time *t*, respectively. The amount of money invested in the risk-free asset at time *t* is $X_t^\pi - b_t$, where X_t^π is the wealth process associated with strategy π.

Definition 1 (*Admissible Strategy*) Let $Q := [0,\mathrm{T}]\times R$. For any fixed $t\in[0,T]$, a strategy π is said to be admissible if it is F_t-progressively measurable, and satisfies that

(i)The no-shorting constraint means that $b_t \ge 0.$ (ii) $0\le a_t \le 1.$ (iii) $E\left(\int_0^T b_t^2\mathrm{d}t\right) < \infty$, for all $T<\infty$.

Denote the set of all admissible strategies by Π. Corresponding to an admissible strategy π and the initial wealth X_0, the wealth process of the insurer can be described as

$$\begin{cases}\mathrm{d}X_t^\pi = \left\{\lambda\mu_2^2(\theta_1-\theta_2 a_t^2) + r_0X_t^\pi + (r_1-r_0)b_t\right\}\mathrm{d}t + \sigma b_t\mathrm{d}W_t + (1-a_t)\sqrt{\lambda}\mu_2\mathrm{d}W_t^{(1)} + \sigma_0\mathrm{d}W_t^{(0)}, \\ X_0^\pi = X_0\end{cases} \quad (2)$$

for the U-S case, or

$$\begin{cases}\mathrm{d}\hat{X}_t^\pi = \left\{(\lambda\mu_2^2+\sigma_0^2)(\theta_1-\theta_2 a_t^2) + r_0\hat{X}_t^\pi + (r_1-r_0)b_t\right\}\mathrm{d}t + \sigma b_t\mathrm{d}W_t + (1-a_t)\sqrt{\lambda}\mu_2\mathrm{d}W_t^{(1)} + \sigma_0(1-a_t)\mathrm{d}W_t^{(0)}, \\ \hat{X}_0^\pi = X_0\end{cases} \quad (3)$$

for the A-C case.

III. Mean-variance problem

In this section, we consider the mean-variance criterion. We aim at finding an admissible strategy such that the expected terminal wealth satisfies $EX_T^\pi = k$ for a given $k\in R$ while the risk measured by the variance of the terminal wealth $VarX_T^\pi = E[X_T^\pi - k]^2$ is minimized for the U-S case, or $E\hat{X}_T^\pi = k$ while $Var\hat{X}_T^\pi = E[\hat{X}_T^\pi - k]^2$ is minimized for the A-C case. Here, we impose

$$k \ge X_0 e^{r_0T} + \tfrac{\lambda\mu_2^2\theta_1}{r_0}(e^{r_0T}-1)$$

for the U-S case, and

$$k \ge X_0 e^{r_0T} + \tfrac{(\lambda\mu_2^2+\sigma_0^2)\theta_1}{r_0}(e^{r_0T}-1)$$

for the A-C case, which coincide with the amount that the insurer would earn if all of the initial wealth were invested in the risk-free asset. Thus, the above problems can be formulated as the following constrained stochastic optimization problem,

$$\min_{\pi\in\Pi} \quad E[X_T^\pi - k]^2$$

$$\text{subject to} \begin{cases} EX_T^\pi = k, \\ (X_\cdot^\pi, \pi) \text{ satisfy (2).} \end{cases} \tag{4}$$

or

$$\min_{\pi\in\Pi} \quad E[\hat{X}_T^\pi - k]^2$$

$$\text{subject to} \begin{cases} E\hat{X}_T^\pi = k, \\ (\hat{X}_\cdot^\pi, \pi) \text{ satisfy (3).} \end{cases} \tag{5}$$

Using the well-known Lagrangian duality method transforms the problem (4) and (5) into the following equivalent min-max problems:

$$\max_{l\in R} \min_{\pi\in\Pi} \{E[X_T^\pi - (k-l)]^2 - l^2\},$$

$$\text{subject to } (X_\cdot^\pi, \pi) \text{ satisfy (2)} \tag{6}$$

or

$$\max_{l\in R} \min_{\pi\in\Pi} \{E[\hat{X}_T^\pi - (k-l)]^2 - l^2\},$$

$$\text{subject to } (\hat{X}_\cdot^\pi, \pi) \text{ satisfy (3)} \tag{7}$$

To solve the problems (4) and (5), or the equivalent min-max problems (6) and (7), we first consider the following unconstrained stochastic optimization problems

$$\min_{\pi\in\Pi} E[X_T^\pi - (k-l)]^2,$$

$$\text{subject to } (X_\cdot^\pi, \pi) \text{ satisfy (2)} \tag{8}$$

or

$$\min_{\pi\in\Pi} E[\hat{X}_T^\pi - (k-l)]^2,$$

$$\text{subject to } (\hat{X}_\cdot^\pi, \pi) \text{ satisfy (3)} \tag{9}$$

Let $x_t^\pi = X_t^\pi - (k-l)$ and $\hat{x}_t^\pi = \hat{X}_t^\pi - (k-l)$. For the auxiliary problems (8) and (9), we define the associated optimal value function using the same notation $J(t,x)$

$$J(t,x) = \min_{\pi\in\Pi} E\left\{\frac{1}{2}[x_T^\pi]^2 \,\middle|\, x_t^\pi = x\right\}$$

or

$$J(t,x) = \min_{\pi\in\Pi} E\left\{\frac{1}{2}[\hat{x}_T^\pi]^2 \,\middle|\, \hat{x}_t^\pi = x\right\}.$$

It is obvious that (2) and (3) can be rewritten as

$$\mathrm{d}x_t^\pi = \left\{\lambda\mu_2^2(\theta_1 - \theta_2 a_t^2) + r_0 x_t^\pi + r_0(k-l) + (r_1 - r_0)b_t\right\}\mathrm{d}t + \sigma b_t \mathrm{d}W_t + (1-a_t)\sqrt{\lambda}\mu_2 \mathrm{d}W_t^{(1)} + \sigma_0 \mathrm{d}W_t^{(0)},$$

$$\mathrm{d}\hat{X}_t^\pi = \left\{(\lambda\mu_2^2 + \sigma_0^2)(\theta_1 - \theta_2 a_t^2) + r_0\hat{X}_t + r_0(k+l) - (r_1 - r_0)b_t\right\}\mathrm{d}t + \sigma b_t \mathrm{d}W_t + (1-a_t)\sqrt{\lambda}\mu_2 \mathrm{d}W_t^{(1)} + \sigma_0 \mathrm{d}W_t^{(0)} \quad (1$$

For function $\phi(t,x)$, let $C^{1,2}$ denote the space of $\phi(t,x)$ such that ϕ and its partial derivatives $\phi_t, \phi_x, \phi_{xx}$ are continuous on $[0,T]\times R.$ To solve the above problem, we use the dynamic programming approach described in Fleming and Soner (1993). From standard arguments, we see that if the value function $J(t,x)\in C^{1,2}$, then $J(t,x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation, for $t<T$,

$$\min_{\pi\in\Pi} A^{\pi} J(t,x)=0, \qquad (10)$$

with the boundary condition

$$J(T,x)=\frac{1}{2}x^2. \qquad (11)$$

where A^{π} denotes the generator of the surplus process controlled by the control variable π. Further, applying Ito's formula for the jump-diffusion process to $J(t,x)$, we deduce that

$A^{\pi}J(t,x)$ in (10) is

$$A^{\pi}J(t,x)=J_t+\left[\lambda\mu_2^2(\theta_1-\theta_2 a_t^2)+r_0 x+r_0(k-l)+(r_1-r_0)b_t\right]J_x+\frac{1}{2}(\sigma^2 b_t^2+\sigma_0^2+\lambda\mu_2^2(1-a_t)^2)J_{xx}$$ for

the U-S case, or

$$A^{\pi}J(t,x)=J_t+\left[(\lambda\mu_2^2+\sigma_0^2)(\theta_1-\theta_2 a_t^2)+r_0 x+r_0(k-l)+(r_1-r_0)b_t\right]J_x+\frac{1}{2}\left(\sigma^2 b_t^2+(\lambda\mu_2^2+\sigma_0^2)(1-a_t)^2\right)J_{xx}$$ for the

A-C case.

By the standard method used in Fleming and Soner (1993), we have the following verification theorem.

Theorem 1 Let $V(t,x)\in C^{1,2}$ be a classical solution to (10) that satisfies (11). Then, the value function $J(t,x)$ coincides with $V(t,x)$, that is, $V(t,x)=J(t,x)$. Furthermore, let $\pi^*=(a_t^*,b_t^*)\in\Pi$ such that $A^{\pi^*}J(t,x)=0$ holds for all $(t,x)\in[0,T]\times R$. Then, the strategy $\pi^*=(a^*_{X_t^*},b^*_{X_t^*})$ is optimal, where X_t^* is the reserve process under the optimal strategy.

IV. Optimal results for the U-S case

In this section, we derive the optimal control strategy and the optimal value function for the U-S case. First of all, we consider the unconstrained stochastic optimization problem (8). Suppose that there exists a function $V(t,x)$ satisfying the conditions given in Theorem 1. According to Theorem 1, the HJB equation (10) is

$$\min_{\pi\in\Pi}\{V_t+\left[\lambda\mu_2^2(\theta_1-\theta_2 a_t^2)+r_0 x+r_0(k-l)+(r_1-r_0)b_t\right]V_x+\frac{1}{2}(\sigma^2 b_t^2+\sigma_0^2+\lambda\mu_2^2(1-a_t)^2)V_{xx}\}=0, \qquad (12)$$

with the boundary condition $V(T,x)=\frac{1}{2}x^2$. According to the structure of (12) and its boundary condition, it is natural to assume

$$V(t,x)=\frac{1}{2}P(t)x^2+W(t)x+K(t). \qquad (13)$$

The boundary condition $V(T,x)=\frac{1}{2}x^2$ implies $P(T)=1, W(T)=0, K(T)=0.$ Inserting (13) into (12) and rearranging results in

$$\min_{\pi\in\Pi}\{\frac{1}{2}P_t(t)x^2+W_t(t)x+K_t(t)+\left[\lambda\mu_2^2(\theta_1-\theta_2 a_t^2)+r_0 x+r_0(k-l)+(r_1-r_0)b_t\right][P(t)x+W(t)] \\ +\frac{1}{2}(\sigma^2 b_t^2+\sigma_0^2+\lambda\mu_2^2(1-a_t)^2)P(t)\}=0 \tag{14}$$

For a_t and b_t without restriction, (14) attains its minimum at

$$a_{t1}=\left(1-2\theta_2(x+\tfrac{W(t)}{P(t)})\right)^{-1},\quad b_{t1}=-\frac{r_1-r_0}{\sigma^2}\left(x+\tfrac{W(t)}{P(t)}\right) \tag{15}$$

Since $b_t\ge 0$ and $0\le a_t\le 1$, if the right of a_{t1} or b_{t1} is less than 0, then we have to truncate it by 0, if the right of a_{t1} is more than 1or goes to infinity, we have to truncate it by 1. So we need to define the following regions:

$$A_1=\{(t,x)\in[0,T]\times R, x+\tfrac{W(t)}{P(t)}<0\},$$
$$A_2=\{(t,x)\in[0,T]\times R, x+\tfrac{W(t)}{P(t)}>\tfrac{1}{2\theta_2}\},$$
$$A_3=\{(t,x)\in[0,T]\times R, 0\le x+\tfrac{W(t)}{P(t)}\le\tfrac{1}{2\theta_2}\}.$$

In the following, we will specify A_1, A_2 and A_3. We first analyze A_1. In this case, the minimum of (14) is attained for

$$a_t^*=a_{t1},\quad b_t^*=b_{t1}. \tag{16}$$

Inserting (16) into (14) and letting $P(t)=P_1(t), W(t)=W_1(t), K(t)=K_1(t)$ in (14) yields a new equation, then multiplying this new equation by $\left(1-2\theta_2(x+\tfrac{W(t)}{P(t)})\right)^2$ and letting the coefficients of x^4, x^3 and x^2 be zero respectively, according to the boundary condition, we finally obtain

$$P_1(t)=e^{(B-2r_0)(t-T)},\ W_1(t)=\frac{m}{r_0}e^{(B-2r_0)(t-T)}\left(1-e^{r_0(t-T)}\right),$$
$$K_1(t)=\frac{m^2}{2r_0^2}e^{B(t-T)}\left(e^{-r_0(t-T)}-1\right)^2-\frac{\lambda\mu_2^2+\sigma_0^2}{2(B-2r_0)}\left(e^{(B-2r_0)(t-T)}-1\right) \tag{17}$$

where

$$B=\frac{(r_1-r_0)^2}{\sigma^2},\quad m=r_0(k-l)+\lambda\theta_1\mu_2^2.$$

Substituting (17) into (15), we have

$$a_t^*=\left(1-2\theta_2 x+\tfrac{2\theta_2 m}{r_0}(e^{r_0(t-T)}-1)\right)^{-1},\quad b_t^*=-\tfrac{r_1-r_0}{\sigma^2}\left(x-\tfrac{m}{r_0}(e^{r_0(t-T)}-1)\right). \tag{18}$$

Meanwhile, we get the solution of HJB equation (12) is

$$V_1(t,x)=\frac{1}{2}e^{(B-2r_0)(t-T)}\left\{x+\frac{m}{r_0}\left(1-e^{r_0(t-T)}\right)\right\}^2-\frac{\lambda\mu_2^2+\sigma_0^2}{2(B-2r_0)}\left(e^{(B-2r_0)(t-T)}-1\right) \tag{19}$$

Replace $P(t), W(t)$ and $K(t)$ by $P_1(t), W_1(t)$ and $K_1(t)$ in (13) results in

$$A_1=\{(t,x)\in[0,T)\times R, x+\frac{m}{r_0}\left(1-e^{r_0(t-T)}\right)<0\}.$$

When $(t,x)\in A_2$, by (15), we obtain $(a_t^*, b_t^*)=(0,0)$, which implies that one invests all the wealth in the risk-free asset and not purchase reinsurance. Similarly, we can obtain the solution of (14)

$$V_2(t,x)=\frac{1}{2}P_2(t)x^2+W_2(t)x+K_2(t),$$

where

$$P_2(t)=e^{-2r_0(t-T)},\quad W_2(t)=\frac{m}{r_0}e^{-2r_0(t-T)}\left(1-e^{r_0(t-T)}\right),\ K_2(t)=\frac{m^2}{2r_0^2}\left(e^{-r_0(t-T)}-1\right)^2+\frac{\lambda\mu_2^2+\sigma_0^2}{4r_0}\left(e^{-2r_0(t-T)}-1\right)$$

That is,

$$V_2(t,x)=\frac{1}{2}e^{-2r_0(t-T)}\left\{x+\frac{m}{r_0}\left(1-e^{r_0(t-T)}\right)\right\}^2+\frac{\lambda\mu_2^2+\sigma_0^2}{4r_0}\left(e^{-2r_0(t-T)}-1\right) \tag{20}$$

Then A_2 can be specified and

$$A_2=\{(t,x)\in[0,T]\times R, x+\frac{m}{r_0}\left(1-e^{r_0(t-T)}\right)>\frac{1}{2\theta_2}\}.$$

Similarly, when $(t,x)\in A_3$, by (15), we have $(a_t^*,b_t^*)=(1,0)$. The solution of (14) is

$$\begin{aligned}V_3(t,x)&=\frac{1}{2}P_3(t)x^2+W_3(t)x+K_3(t)\\&=\frac{1}{2}e^{-2r_0(t-T)}\left\{x+\frac{m-\lambda\mu_2^2\theta_2}{r_0}\left(1-e^{r_0(t-T)}\right)\right\}^2+\frac{\sigma_0^2}{4r_0}\left(e^{-2r_0(t-T)}-1\right),\end{aligned} \tag{21}$$

where

$$P_3(t)=e^{-2r_0(t-T)},\quad W_3(t)=\frac{m-\lambda\mu_2^2\theta_2}{r_0}e^{-2r_0(t-T)}\left(1-e^{r_0(t-T)}\right),$$

$$K_3(t)=\frac{1}{2}\left(\frac{m-\lambda\mu_2^2\theta_2}{r_0}\right)^2\left(e^{-r_0(t-T)}-1\right)^2+\frac{\sigma_0^2}{4r_0}\left(e^{-2r_0(t-T)}-1\right)$$

Thus A_3 can be rewritten as

$$A_3=\{(t,x)\in[0,T]\times R, 0\le x+\frac{m-\lambda\mu_2^2\theta_2}{r_0}\left(1-e^{r_0(t-T)}\right)\le\tfrac{1}{2\theta_2}\}.$$

For the U-S case, all above derivations are summarized by the theorem below.

Theorem 2 For the unconstrained stochastic optimization problem (8), the optimal control strategy (a_t^*,b_t^*) and the optimal value function $V(t,x)$ are given as follows:

(1)If $x+\frac{m}{r_0}\left(1-e^{r_0(t-T)}\right)<0$, (a_t^*,b_t^*) is given by (18) and $V(t,x)=V_1(t,x)$ which is given by (19);

(2)If $x+\frac{m}{r_0}\left(1-e^{r_0(t-T)}\right)>\frac{1}{2\theta_2}$, $(a_t^*,b_t^*)=(0,0)$ and $V(t,x)=V_2(t,x)$ given by (20);

(3)If $0\le x+\frac{m-\lambda\mu_2^2\theta_2}{r_0}\left(1-e^{r_0(t-T)}\right)\le\frac{1}{2\theta_2}$, $(a_t^*,b_t^*)=(1,0)$ and $V(t,x)=V_3(t,x)$, see (21).

Next, we try to solve the mean-variance problem (4) which can be transformed into the equivalent min-max problems (6). Let $t=0, x=x_0=X_0-k+l$ in Theorem 2.

When $x_0+\frac{m}{r_0}\left(1-e^{-r_0T}\right)<0$, that is,

$$X_0e^{r_0T}+l-k+\frac{\lambda\theta_1\mu_2^2}{r_0}(e^{r_0T}-1)<0,$$

since

$$\begin{aligned}\min_{\pi\in\Pi}E[\frac{1}{2}(X_T^\pi-k)^2]&=\min_{\pi\in\Pi}\left\{E[\frac{1}{2}(X_T^\pi-k)^2]+lE[X_T^\pi-k]\right\}=\min_{\pi\in\Pi}\left\{E[\frac{1}{2}(X_T^\pi+l-k)^2]-\frac{1}{2}l^2\right\}\\&=\min_{\pi\in\Pi}\left\{E\left(\frac{1}{2}[x_T^\pi]^2\right)-\frac{1}{2}l^2\right\}=V_1(0,x_0)-\frac{1}{2}l^2,\end{aligned}$$

for a fixed l, where $V_1(0,x_0)$ can by expressed by (19). We denote

$$W_1(l)=\min_{\pi\in\Pi}VarX_T^\pi=\min_{\pi\in\Pi}\left\{E[X_T^\pi-k]^2+2lE[X_T^\pi-k]\right\}=2V_1(0,x_0)-l^2.$$

To obtain the optimal strategy and the optimal value function for original problem (4), one needs to maximize $W_1(l)$ over $l\in R$ according to the Lagrange duality theorem. Differentiating $W_1(l)$ with respect to l, and setting the derivative equal to zero, we get

$$l^*=\frac{X_0e^{r_0T}-k-\frac{\lambda\theta_1\mu_2^2}{r_0}(1-e^{r_0T})}{e^{BT}-1}. \tag{22}$$

A simple calculation shows that $W_1(l)$ attains its maximum value

$$W_1(l^*)=\min_{\pi\in\Pi}VarX_T^\pi=\frac{[X_0e^{r_0T}-k-\frac{\lambda\theta_1\mu_2^2}{r_0}(1-e^{r_0T})]^2}{e^{BT}-1}+\frac{\lambda\mu_2^2+\sigma_0^2}{B-2r_0}\left(1-e^{-(B-2r_0)T}\right), \tag{23}$$

at l^*, where we have used the fact that $k\ge X_0e^{r_0t}+\frac{\lambda\theta_1\mu_2^2}{r_0}(e^{r_0t}-1)$, due to the previous assumption.

Similarly, if

$$X_0e^{r_0T}+l-k+\frac{\lambda\theta_1\mu_2^2}{r_0}(e^{r_0T}-1)>\frac{e^{r_0T}}{2\theta_2},$$

it is easily verified that there isn't an optimal value function.

If

$$0\le X_0e^{r_0T}+\frac{\lambda\mu_2^2(\theta_1-\theta_2)}{r_0}\left(e^{r_0T}-1\right)+l-k\le\frac{e^{r_0T}}{2\theta_2},$$

for a fixed l, denote

$$W_3(l)=\min_{\pi\in\Pi}VarX_T^\pi=\min_{\pi\in\Pi}E[X_T^\pi-k]^2+2lE[X_T^\pi-k]=2V_3(0,x_0)-l^2,$$

where $V_3(0,x_0)$ is given by (21). It is not difficult to find that $W_3(l)$ decreasing with respect to l attains its maximum value

$$\max_{l\in R}W_3(l)=\min_{\pi\in\Pi}VarX_T^\pi=\frac{\sigma_0^2}{2r_0}\left(e^{2r_0T}-1\right)-\left(k-X_0e^{r_0T}-\frac{\lambda\mu_2^2(\theta_1-\theta_2)}{r_0}\left(e^{r_0T}-1\right)\right)^2. \tag{24}$$

at

$$l=k-X_0e^{r_0T}-\frac{\lambda\mu_2^2(\theta_1-\theta_2)}{r_0}\left(e^{r_0T}-1\right).$$

The above results are summarized in the following theorem.

Theorem 3 For problem (4) with the expected terminal wealth $X_T^\pi=k$,

(1)If $X_t^\pi e^{-r_0(t-T)}+l^*-k+\dfrac{\lambda\theta_1\mu_2^2}{r_0}(e^{-r_0(t-T)}-1)<0$, the optimal investment-reinsurance strategy is

$$a_t^*=\left(1-2\theta_2l^*e^{-(B-r_0)(t-T)}\right)^{-1},\quad b_t^*=-Bl^*e^{-(B-r_0)(t-T)},$$

and the efficient frontier is

$$\min_{\pi\in\Pi}VarX_T^\pi=\frac{[X_0e^{r_0T}-k-\frac{\lambda\theta_1\mu_2^2}{r_0}(1-e^{r_0T})]^2}{e^{BT}-1}+\frac{\lambda\mu_2^2+\sigma_0^2}{B-2r_0}\left(1-e^{-(B-2r_0)T}\right).$$

(2)If $X_t^\pi e^{-r_0(t-T)}+l-k+\dfrac{\lambda\theta_1\mu_2^2}{r_0}(e^{-r_0(t-T)}-1)>\dfrac{e^{-r_0(t-T)}}{2\theta_2}$, the optimal investment-reinsurance strategy is $(a_t^*,b_t^*)=(0,0)$, but there is no efficient frontier.

(3)If $0\le X_t^\pi e^{-r_0(t-T)}+l-k+\dfrac{\lambda\mu_2^2(\theta_1-\theta_2)}{r_0}\left(e^{-r_0(t-T)}-1\right)\le\dfrac{e^{-r_0(t-T)}}{2\theta_2}$, the optimal investment-reinsurance strategy is $(a_t^*,b_t^*)=(1,0)$ and the efficient frontier is

$$\min_{\pi\in\Pi} VarX_T^{\pi} = \frac{\sigma_0^2}{2r_0}\left(e^{2r_0T}-1\right)-\left(k-X_0e^{r_0T}-\frac{\lambda\mu_2^2(\theta_1-\theta_2)}{r_0}\left(e^{r_0T}-1\right)\right)^2.$$

V. Optimal results for the A-C case

In this section, we consider the optimal control strategy and the optimal value function for the A-C case. To solve the mean-variance problem (5), we first solve the unconstrained stochastic optimization problem (9). Suppose that there exists a function $\hat{V}(t,x)$ satisfying the conditions given in Theorem 1. According to Theorem 1, for the A-C case, the HJB equation (10) is

$$\min_{\pi\in\Pi}\{\hat{V}_t+\left[(\lambda\mu_2^2+\sigma_0^2)(\theta_1-\theta_2a_t^2)+r_0x+r_0(k-l)+(r_1-r_0)b_t\right]\hat{V}_x+\frac{1}{2}(\sigma^2b_t^2+(\lambda\mu_2^2+\sigma_0^2)(1-a_t)^2)\hat{V}_{xx}\}=0, \quad (25)$$

with the boundary condition $\hat{V}(T,x)=\frac{1}{2}x^2.$

According to the structure of (25) and its boundary condition, it is natural to assume

$$\hat{V}(t,x)=\frac{1}{2}\hat{P}(t)x^2+\hat{W}(t)x+\hat{K}(t). \quad (26)$$

The boundary condition implies $\hat{P}(T)=1,\hat{W}(T)=0,\hat{K}(T)=0.$ Inserting (26) into (25) and rearranging results in

$$\begin{aligned}&\min_{\pi\in\Pi}\{\frac{1}{2}\hat{P}_t(t)x^2+\hat{W}_t(t)x+\hat{K}_t(t)+\left[(\lambda\mu_2^2+\sigma_0^2)(\theta_1-\theta_2a_t^2)+r_0x+r_0(k-l)+(r_1-r_0)b_t\right][\hat{P}(t)x+\hat{W}(t)]\\&+\frac{1}{2}\left(\sigma^2b_t^2+(\lambda\mu_2^2+\sigma_0^2)(1-a_t)^2\right)\hat{P}(t)\}=0.\end{aligned} \quad (27)$$

For a_t and b_t without restriction, (27) attains its minimum at

$$a_{t2}=\left(1-2\theta_2(x+\tfrac{\hat{W}(t)}{\hat{P}(t)})\right)^{-1},\quad b_{t2}=-\tfrac{r_1-r_0}{\sigma^2}\left(x+\tfrac{\hat{W}(t)}{\hat{P}(t)}\right)$$

If the right of a_{t2} or b_{t2} is less than 0, then we have to truncate it by 0, if the right of a_{t2} is more than 1 or goes to infinity, we have to truncate it by 1.

Similar to the discussion of the Theorem 2, for the A-C case, we can derive the optimal control strategy and the optimal value function for the unconstrained stochastic optimization problem (9). These results are summarized by the theorem below. Let $\hat{m}=(\lambda\mu_2^2+\sigma_0^2)\theta_1+r_0(k-l).$

Theorem 4 For the unconstrained stochastic optimization problem (9),

(1) If $x+\frac{\hat{m}}{r_0}\left(1-e^{r_0(t-T)}\right)<0$, the optimal control strategy is (a_t^*,b_t^*), where

$$a_t^*=\left(1-2\theta_2(x+\frac{\hat{m}}{r_0}(1-e^{r_0(t-T)}))\right)^{-1},\quad b_t^*=-\frac{r_1-r_0}{\sigma^2}\left(x+\frac{\hat{m}}{r_0}(1-e^{r_0(t-T)})\right).$$

and the optimal value function is $\hat{V}_1(t,x)=V_1(t,x),$ where $V_1(t,x)$ is given by (19) .

(2) If $x+\frac{\hat{m}}{r_0}\left(1-e^{r_0(t-T)}\right)>\frac{1}{2\theta_2}$, the optimal control strategy is $(a_t^*,b_t^*)=(0,0)$, and the optimal value function is

$$\hat{V}_2(t,x)=\frac{1}{2}e^{-2r_0(t-T)}\left(x+\frac{\hat{m}}{r_0}(1-e^{r_0(t-T)})\right)^2+\frac{\lambda\mu_2^2+\sigma_0^2}{4r_0}\left(e^{-2r_0(t-T)}-1\right).$$

(3) If $0\le x+\frac{\hat{m}-\lambda\mu_2^2\theta_2}{r_0}\left(1-e^{r_0(t-T)}\right)\le\frac{1}{2\theta_2}$, the optimal control strategy is $(a_t^*,b_t^*)=(1,0)$, and the optimal value function is

$$\hat{V}_3(t,x)=\frac{1}{2}e^{-2r_0(t-T)}\left(x+\frac{\hat{m}-\lambda\mu_2^2\theta_2}{r_0}\left(1-e^{r_0(t-T)}\right)\right)^2.$$

For the A-C case, using a similar argument to **Theorem 3,** we find that there are an optimal strategy and an optimal value function for original mean-variance problem (5) only if

$$\hat{X}_t^{\pi}e^{-r_0(t-T)}+\frac{(\lambda\mu_2^2+\sigma_0^2)\theta_1}{r_0}(e^{-r_0(t-T)}-1)-k+\hat{l}^*<0,$$

which are summarized in the following theorem 5. Here, we have

$$\hat{l}^*=\frac{X_0e^{r_0T}-k-\dfrac{(\lambda\mu_2^2+\sigma_0^2)\theta_1}{r_0}(1-e^{r_0T})}{e^{BT}-1}.$$

Theorem 5 For the mean-variance problem (5) with the expected terminal wealth $\hat{X}_T^{\pi}=k,$ if

$$\hat{X}_t^{\pi}e^{-r_0(t-T)}+\frac{(\lambda\mu_2^2+\sigma_0^2)\theta_1}{r_0}(e^{-r_0(t-T)}-1)-k+\hat{l}^*<0,$$

the optimal invest-reinsurance strategy is (a_t^*,b_t^*), where

$$a_t^*=\left(1-2\theta_2\hat{l}^*e^{-(B-r_0)(t-T)}\right)^{-1},\quad b_t^*=-B\hat{l}^*e^{-(B-r_0)(t-T)}.$$

The efficient frontier is

$$\min_{\pi\in\Pi}Var\hat{X}_T^{\pi}=\frac{[X_0e^{r_0T}+\dfrac{(\lambda\mu_2^2+\sigma_0^2)\theta_1}{r_0}(e^{r_0T}-1)-k]^2}{e^{BT}-1}+\frac{\lambda\mu_2^2+\sigma_0^2}{B-2r_0}\left(1-e^{-(B-2r_0)T}\right).$$

Comparing the above results, we find that the optimal reinsurance strategies and the optimal investment strategies are different for the U-S case and the A-C case.

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