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The Paradox of Sharpe ratio: a modified Sharpe ratio

Kun Xiao[1], Peibiao Zhao[2]
[1]*(School of Science, Nanjing University of Science and Technology, China)*
[2]*(School of Science, Nanjing University of Science and Technology, China)*

Abstract : *The paradox of Sharpe ratio, roughly speaking, the conclusions generated by Sharpe ratio are inconsistent with those induced by the random dominance theory, is an open problem. This paper investigates and achieves a modified Sharpe ratio via the probability semimeasures, which gives a positive answer to the so-called Sharpe ratio paradox. In addition, we also consider and modify the Sharpe ratio via the sublinear expectation under the sense of uncertain distributions.*

Keywords - *Fund performance measurement， G-expectation， Monotonicity， Sharpe ratio， Random dominance， Risk measurement.*

I. Introduction

The Sharpe Ratio, also known as the Sharpe Index, is a well-known indicator for measuring fund returns. We know that it is not enough to invest solely in return, but also in terms of risk tolerated, that is, return to risk ratio. The Sharpe ratio describes this concept, that is, for every unit of total risk, how much excess returns will be produced. The Sharpe ratio is currently widely used in the industry and is one of the important indicators of fund performance measurement. However, the study found that the Sharpe ratio has some shortcomings, such as failing to consider the fact that the actual rate of return data may have sharp peaks, thick tails and biased characteristics, and only using the mean and variance to describe the distribution characteristics of the rate of return. Based on this, many scholars continue to revise the Sharpe ratio, and the revision process generally moves in the following two directions.

The first revision idea is to relax the assumption of normal distribution and find other distributions to capture the peak, thick tail and biased characteristics of the actual rate of return series. For example, Shi, Wang and Xu [1] used an asymmetric Laplace distribution to fit the distribution of returns. They believed that this distribution took into account the bias and thick tails of the return distribution and was better than the normal distribution. Liu and Chen [2] believe that the Levy Tempered Stable distribution can better describe the characteristics of financial asset yield. Yu, Tian and Wang [3] proposed to use the Skewed-t distribution to fit the distribution of fund returns. In short, the research in this area assumes a specific distribution function in advance, then uses historical financial data to estimate the undetermined parameters in it, and modifies the Sharpe ratio based on this. We know that there are many distribution functions that can theoretically describe the characteristics of sharp peaks, thick tails and biased characteristics, but which distribution is most suitable for fitting the true distribution of returns lacks a solid theoretical basis for testing. In the actual financial market, investors do not know the form of the true distribution function. If a specific distribution function is artificially set, it is easy to cause model setting errors.

The second direction of revision is to find risk indicators that can reflect the characteristics of sharp peaks, thick tails and biases to replace the standard deviation indicators to modify the Sharpe ratio. This research has got rid of the dependence on the form of the characteristic distribution function, but has many requirements on the nature of the selected risk indicators. Among the most influential works are the research by Sortino and Price [4] and Alexander and Baptista [5], Sortino and Price [4] use the downside standard deviation instead of the standard deviation in the classic Sharpe ratio to correct them. Alexander and Baptista [5] used VaR instead of standard deviation to establish a modified Sharpe ratio. Both VaR and the standard deviation below can reflect the information of higher moments, and these higher moments can characterize the peak, thick tail and biased characteristics of asset returns. Unfortunately, these two indicators are not a coherent risk measurement. Later, Huang, Li and Ding [6] used $CVaR$ to replace the standard deviation in the Sharpe ratio, and found that the Sharpe ratio corrected by $CVaR$ has many excellent properties. In short, finding other more reasonable risk indicators that can reflect higher moment information to revise the Sharpe ratio is worthy of further research in the future. In fact, one can try to modify the Sharpe rate via any other risk measures such as coherent risk measures [7], convex risk measures [8] and quasi-convex risk measures (Non-convex Non-cash Risk Measures [9]) following this argument.

II. Preliminaries

At present, the theory of random dominance has been widely used in the study of economic problems, and the theory of random dominance is developed on the basis of the expected utility theory. It uses part of the information of the utility function, but does not require a specific form of the utility function. Suppose X and Y are two random variables, and their distribution functions are F and G, then the definitions of first-order random dominance and second-order random dominance are as follows.

Definition 2.1 If for any x, there is $F(x) \leq G(x)$, and there is at least one x_0 with $F(x_0) \leq G(x_0)$, then X is called first-order random to be better than Y, denoted as $X \succ_{FSD} Y$.

Definition 2.2 If for any x, there is always $\int_{-\infty}^{x}[F(t)-G(t)]dt \leq 0$,and there is at least one x_0 with $\int_{-\infty}^{x_0}[F(t)-G(t)]dt \leq 0$,then X is called second-order random to be better than Y, denoted as $X \succ_{SSD} Y$.

Definition 2.3 randomly dominant monotonicity: for any rate of return variable r_A, r_B, If r_A first-order (or second-order) dominates r_B randomly, the monotonicity of random dominance requires ρ to satisfy: $\rho(r_A) \leq \rho(r_B)$.

Let $\tilde{r}$ be the rate of return of risky assets in the financial market, r_f be the rate of return of risk-free assets, then the excess rate of return of risky assets $r = \tilde{r} - r_f$,the definition of Sharpe ratio is:

$$SR(r) = \frac{E(r)}{\sqrt{Var(r)}} = \frac{E(\tilde{r})-r_f}{\sqrt{Var(\tilde{r})}} \tag{1}$$

Where $E(\cdot)$ and $Var(\cdot)$ are the expectation and variance operations respectively. The definition of the Sharpe ratio is how much excess returns will be generated for each unit of total risk that the portfolio is exposed to. The higher the Sharpe ratio, the greater the value of the investment. The Sharpe ratio uses the standard deviation as a risk measurement index, which implies the assumption that the actual rate of return obeys a normal distribution. It does not take into account that the data may exhibit non-normal distribution characteristics such as sharp peaks and thick tails and asymmetry. In addition, the Sharpe ratio may produce conclusions that are inconsistent with the random dominance theory, leading to the Sharpe ratio paradox. For example: Suppose there are two assets named A and B respectively. Let their excess return rates be respectively r_A, r_B, and the distribution of r_A has three results: -2%, 2 % and 8 %, and the corresponding probabilities are 0.2, 0.2 and 0.4, the distribution results of r_B are -2 %, 2 % and 6 %, and the corresponding probabilities are 0.2, 0.4 and 0.4. It can be easily verified that A first-order randomly dominates B, so rational investors will prefer asset A more. But through calculation, it can be found that $SR(A) \approx 0.928 < SR(B) \approx 0.935$, that is to say, asset B is more worthy of investment for everyone than asset A, then the problem arises: This conflicts with rational investor preferences. The reason for the contradiction is actually that the standard deviation index in the Sharpe ratio does not have the monotonicity of random dominance, which leads to inconsistency with the theory of random dominance. In this example, if the standard deviation is used to evaluate the risk, the standard deviation of asset A is calculated to be greater than B, that is, the risk of asset A is greater than that of asset B, but asset A randomly dominates asset B in the first order, which runs counter to the random theory of worry.

III. Sharpe ratio based on a new risk measure

wen and Qin [10] proposed a new risk measure based on the probability semi-metric, and found that the risk measure satisfies convexity and random dominance monotonicity, and it contains many existing risk measurement methods, which are defined as follows:

$$\rho_S(X) = E(H(R-X)^+) \tag{2}$$

here $H(x)$ is a monotonically increasing non-negative convex function defined on $\mathbb{R}^+$, R is a reference level, $(R-X)^+ = max(R-X, 0)$.

Based on the above idea of using VaR and $CVaR$ to replace the standard deviation in the Sharpe ratio to correct the Sharpe ratio, this paper decides to replace the standard deviation in the classic Sharpe ratio with ρ_S, and given a new fund performance measure PM, defined as follows:

$$PM(r) = \frac{E(r)}{E(H(R-X)^+)} \tag{3}$$

The study found that PM indicators have some good properties as follows:

Proposition 3.1 PM is monotonic. For any rate of return variable r_A, r_B , if $r_A \le r_B$, then $PM(r_A) \le PM(r_B)$.That is, if the rate of return of asset A is greater than asset B at any time, asset A performs better than asset B under the PM indicator.

Proof If $r_A \le r_B$, there is $(R - r_A)^+ \ge (R - r_B)^+$ and $Er_A \le Er_B$. We know that $H(x)$ is monotonically increasing, so there is $H(R - r_A)^+ \ge H(R - r_B)^+$ .Then we have $E[H(R - r_A)^+] \ge E[H(R - r_B)^+]$,that is, $PM(r_A) \le PM(r_B)$. □

Proposition 3.2 The PM value of the asset portfolio is greater than the minimum of the PM values of all individual assets. That is, for any rate of return variable r_A, r_B, and $\lambda \in [0,1]$, we can prove that the following formula is valid:$PM\big(\lambda r_A + (1 - \lambda r_B)\big) \ge min\{PM(r_A), PM(r_B)\}$.

Proof If $r_A \ge r_B$, there is $\lambda r_A + (1-\lambda) r_B \ge r_B$, we have $E(\lambda r_A + (1-\lambda) r_B) \ge E(r_B)$ and we have $\rho_S(\lambda r_A + (1-\lambda) r_B) \le \lambda \rho_S(r_A) + (1-\lambda)\rho_S(r_B) \le \lambda \rho_S(r_B) + (1-\lambda)\rho_S(r_B) = \rho_S(r_B)$, that is $PM(\lambda r_A + (1-\lambda) r_B) \ge PM(r_B)$; The same is true when $r_A \le r_B$.

In short, $PM\big(\lambda r_A + (1 - \lambda r_B)\big) \ge min\{PM(r_A), PM(r_B)\}$. □

This property can be explained as investing in high-performance assets can improve the performance of the overall asset portfolio.

Proposition 3.3 Suppose the excess return rates of assets A and B are r_A, r_B,respectively, if $A \succ_{FSD} B$ or $A \succ_{SSD} B$, then $PM(r_A) \ge PM(r_B)$.

Proof If there is $r_A \succ_{FSD} r_B$ or $r_A \succ_{SSD} r_B$, then $Er_A \ge Er_B$; Since ρ_S satisfies random dominant monotonicity and according to definition 2.3 ,when $r_A \succ_{FSD} r_B$ or $r_A \succ_{SSD} r_B$, there is $E[H(R - r_A)^+] \le E[H(R - r_B)^+]$, that is $PM(r_A) \ge PM(r_B)$. □

The PM index satisfies the monotonicity of random dominance, which is consistent with the random dominance theory, thus solving the problem of the Sharpe ratio paradox, and is superior to the classic Sharpe ratio in a theoretical sense.

IV. The correction of Sharpe ratio under uncertainty of distribution

The probability theory is a basic mathematical tool for solving financial risk problems. In the classical probability space, a triple $(\Omega,\ \mathbb{F},\ \mathbb{P})$, where Ω represents all possible random events; $\mathbb{F}$ represents the set of all related random event combi-nations; and $\mathbb{P}$ represents the probability measure of random event combinations. Classic probability theory has been widely used to describe random events in the financial field, but with the rapid changes in the financial market, the uncertainty in the current financial market has greatly increased, and the classical probability theory method is no longer accurate. To solve the problem of risk measurement in the financial market. Different from the classical linear probability system, Peng [11,12] introduced a new nonlinear expectation G-expectation and the associated G-normal distribution in 2006.G-expectation does not depend on a given probability space, and it describes the uncertainty of the variance of random variables, so it can more essentially describe the risk of financial uncertainty.

4.1 A new risk measure in a sublinear space

So far, when dealing with financial risks, there is an assumption: the prior prob-ability distribution of financial risk asset portfolios has been known in advance, that is, the return rate of assets obeys a certain distribution, but in actual financial markets Investors are faced with various risky financial assets with completely uncertain probability distributions. For uncertain financial assets with various risks, it is unreasonable to use the Sharpe ratio to evaluate the performance of the fund at this time, so we introduce a sub-linear expectation space to deal with risks un-der the condition of uncertain distribution. Based on the risk measurement method proposed by Wen and Qin [10] above, we define a new risk measurement method in the sublinear expected space to replace the standard deviation in the Sharpe ratio. In the case of uncertain distribution, a new risk measurement method based on probability semimetric is proposed:

$$\rho(X) = \hat{E}(H(R-X)^+) \tag{4}$$

$H(x)$ is a monotonically increasing function defined on $\mathbb{R}$. $H(0)$ is a non-negative convex function. R can be a constant or a random variable. $\hat{E}$ is G-expectation. We call it GRM. Let's take a look at some of the properties satisfied by GRM.

Proposition 4.1 GRM satisfy monotonicity.

Proof If $r_A \leq r_B$, there is $(R-r_A)^+ \geq (R-r_B)^+$; we know that $H(x)$ is a monotonically increasing function, so there is $H(R-r_A)^+ \geq H(R-r_B)^+$. According to $\hat{E}$ satisfies monotonicity, then we have $\hat{E}[H(R-X)^+] \geq \hat{E}[H(R-Y)^+]$ □

Proposition 4.2 GRM satisfy convexity.

Proof We have that $R-\lambda X-(1-\lambda)Y = \lambda(R-X)+(1-\lambda)(R-Y)$, we also know that $(a+b)^+ \leq a^+ + b^+$, so we have $[R-\lambda X-(1-\lambda)Y]^+ = [\lambda(R-X)+(1-\lambda)(R-Y)]^+ \leq \lambda(R-X)^+ + (1-\lambda)(R-Y)^+$, and $H(x)$ is a monotonically increasing function, so we can get that $\hat{E}[H(R-\lambda X-(1-\lambda)Y)^+] \leq \hat{E}[H(\lambda(R-X)^+ + (1-\lambda)(R-Y)^+)] \leq \hat{E}[\lambda H(R-X)^+ + (1-\lambda)H(R-Y)^+]$,Then according to $\hat{E}X + \hat{E}Y \geq \hat{E}(X+Y)$, we can easily get that:

$$\hat{E}[\lambda H(R-X)^+ + (1-\lambda)H(R-Y)^+] \leq \hat{E}[\lambda H(R-X)^+] + \hat{E}[(1-\lambda)H(R-Y)^+] \tag{5}$$

$$= \lambda\rho(X) + (1-\lambda)\rho(Y) \tag{6}$$

That is, $GRM(\lambda X + (1-\lambda)y) \leq \lambda GRM(X) + (1-\lambda)GRM(Y)$. □

4.2 The Sharpe Ratio Correction Based on GRM

Most of the fund performance measures used to process data so far are based on the assumption that the distribution of fund returns is known or partially known, but it is obviously unreasonable, because in the actual financial market,the rate of return is unclear, and we cannot know the true shape of the distribution. The uncertainty of this distribution will inevitably affect the precise measurement and prevention of risks, so we cannot ignore it. Based on this, inspired by the improvement of the Sharpe ratio by the predecessors, we introduced a sub-linear expectation in the Sharpe ratio， which is to replace the standard deviation indicator in the Sharpe ratio with the above $\rho(X) = \hat{E}[H(R-X)^+]$, and then give the following fund performance measurement(GPM):

$$GPM(r) = \frac{\hat{E}(r)}{\hat{E}[H(R-r)^+]} \tag{7}$$

Proposition 4.3 GPM satisfies monotonicity. For two return rate variables r_A, r_B， if $r_A \leq r_B$,then $GPM(r_A) \leq GPM(r_B)$. That is, if the rate of return of asset A is greater than that of asset B at any time, then the asset A performs better than the asset B under the GPM indicator.

Proof If $r_A \leq r_B$， then there is $\hat{E}r_A \leq \hat{E}r_B$ and $(R-r_A)^+ \geq (R-r_B)^+$; we know that $H(x)$ is a monotonically increasing function, so there is $H(R-r_A)^+ \geq H(R-r_B)^+$, then we can get that

$$\hat{E}[H(R-r_A)^+] \geq \hat{E}[H(R-r_B)^+] \tag{8}$$

that is, $GPM(r_A) \leq GPM(r_B)$. □

Proposition 4.4 For any rate of return variable r_A, r_B, and $\lambda \in [0,1]$, we have that:

$$GPM(\lambda r_A + (1-\lambda) r_B) \geq min\{GPM(r_A), GPM(r_B)\} \quad (9)$$

Proof If $r_A \geq r_B$, there is $\lambda r_A + (1-\lambda) r_B \geq r_B$, we have $\hat{E}(\lambda r_A + (1-\lambda) r_B) \geq \hat{E}(r_B)$; and we have $\rho(\lambda r_A + (1-\lambda) r_B) \leq \lambda\rho(r_A) + (1-\lambda)\rho(r_B) \leq \lambda\rho(r_B) + (1-\lambda)\rho(r_B) = \rho(r_B)$, and the same is true when $r_A \leq r_B$.

So we can get that $GPM(\lambda r_A + (1-\lambda) r_B) \geq min\{GPM(r_A), GPM(r_B)\}$. □

V. Experiment analysis

5.1 Data collection and analysis

The codes of the ten Chinese fund data used in this experiment are (020001), (020002),050001), (070003), (080001), (202202), (090002), (112002), (121001) and (202101). Taking their cumulative unit net value data from January 1, 2018 to December 31,2018, a total of 2484 data are available. In addition, for the risk-free interest rate, this article uses the Bank of China 1-year deposit interest rate of 1.5%,the daily average interest rate is 0.00410959, and the set confidence level is 99%.Let P_t be the daily accumulated unit net value, and logarithmize P_t, to reduce the volatility of the data, then we have $\hat{R} = \ln P_t - \ln P_{t-1}$. Based on this, using Eviews software to obtain the historical return histogram and data statistical analysis results of ten funds, see the following figure, of which 020002 is taken as an example:

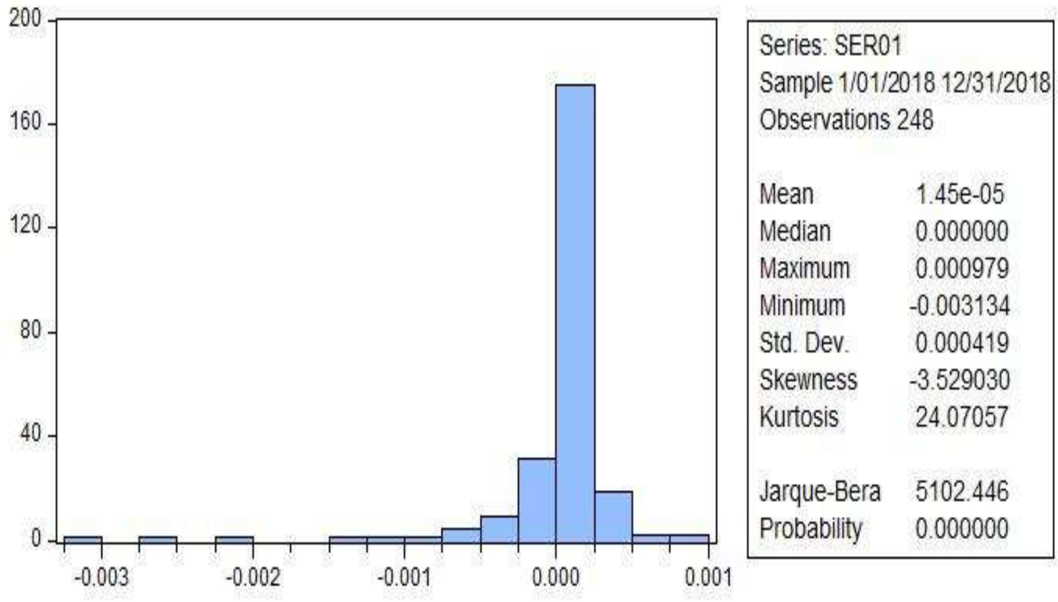

Figure 1: Statistical analysis of fund (020002) data

It can be seen from the Fig1 that the kurtosis is 24.07057, which is greater than the kurtosis of the normal distribution, and is steeper than the peak of the normal distribution, and the skewness of the return series is -3.529030, indicating that there is a serious left It is biased, and the JB statistic is much larger than the critical value. From this， it can be inferred that the yield sequence of 020002 has the characteristics of peak and thick tail, and does not obey the normal distribution, so we cannot use the normal distribution to fit the yield sequence . We divided the ten funds into two groups. The kurtosis and skewness of the five funds in the first group are not much different, and the kurtosis and skewness of the second group are very different, as shown in the following table:

Table 1: The kurtosis and skewness of the first group funds

Fund code	kurtosis	skewness
020001	3.618922	0.115069
050001	3.974652	0.117495
202202	3.95598	0.106843
112002	3.948680	0.169243
202101	3.619456	0.106699

Table 2: The kurtosis and skewness of the second group funds

Fund code	kurtosis	skewness
020002	54.07057	3.52903
070003	3.927867	0.001039
080001	4.86156	0.477466
090002	11.95052	1.83486
121001	5.748544	0.386049

5.2 Comparative analysis between fund performance measurement

This article uses the historical simulation method to analyze and calculate VaR, we set the confidence level to 99%. Since the basic meaning of VaR is the maximum loss that may occur in the future financial assets under a certain degree of confidence and holding period. Therefore, we can estimate the value of VaR with the $n-\alpha nth$ sequential statistics of the excess return rate, that is, $VaR_\alpha = \hat{R}_{n-\alpha n}$. n is the total number of sample data for each fund. We know that $CVaR$ is developed on the basis of VaR, so we plan to use $CVaR_\alpha = VaR_\alpha + \sum_{i=1}^{n-n\alpha} \frac{X_i - VaR_\alpha}{n-\alpha n}$ to estimate $CVaR$ for ten funds. For the $\rho_S(X) = E[H(R-X)^+]$ proposed in the previous article, we intend to take a special value for calculation, so we take $H(x) = x$ and $R = CVaR$. When the confidence level is equal to 99%, we estimate the special value of the above ten funds. Then we use these values to replace the value of the standard deviation in the Sharpe ratio, and we can get three revised Sharpe ratio values. We give the following fund ranking comparison, numbers in brackets indicate ranking.

Table 3: Comparison of the first group fund performance measurement

Fund code	SR_{VaR}	SR_{CVaR}	PM	SR
020001	0.02112(4)	0.01999(4)	0.01978(4)	0.06237(4)
050001	0.05558(1)	0.04460(1)	0.04293(1)	0.09592(1)
202202	0.02538(3)	0.02337(3)	0.02301(3)	0.7978(3)
112002	0.03175(2)	0.03067(2)	0.03031(2)	0.08225(2)
202101	0.00764(5)	0.00764(5)	0.00730(5)	0.02061(5)

Table 4: Comparison of the first group fund performance measurement

Fund code	SR_{VaR}	SR_{CVaR}	PM	SR
020002	0.22437(1)	0.16842(1)	0.16055(1)	0.0926(2)
070003	0.00674(5)	0.00529(5)	0.00508(5)	0.08461(4)
080001	0.02975(3)	0.02092(4)	0.01976(4)	0.03548(5)
090002	0.03099(2)	0.02874(2)	0.02833(2)	0.38597(1)
121001	0.02835(4)	0.02252(3)	0.02175(3)	0.08592(3)

We found that in the comparison of the first group of fund performance measures, the rankings given by the three modified Sharpe ratios are consistent with the fund performance rankings given by the classic Sharpe ratios. In the second set of comparisons, there is a big difference between the three fund performance rankings and the fund performance rankings given by the classic Sharpe ratio. This is because the kurtosis coefficient and the skewness coefficient of the excess re-turns of the first group of funds are not much different， and it is not important whether to consider the higher order moments. Therefore, the difference in the ranking of funds based on the classic Sharpe ratio and the modified Sharpe ratio is basically negligible; while the kurtosis and skewness coefficients of the second group of funds are very different, and the higher moments will affect the calculated values of VaR and $CVaR$, thereby affecting the fund performance ranking, so the fund performance measurement that takes into account the higher moment information gives a more reasonable fund ranking. However, because VaR does not satisfy the monotonicity of second-order random dominance, the fund ranking based on $CVaR$ and $\rho_S(X)'s$ fund performance measurement is more reasonable than the fund ranking based on VaR's fund performance measurement.

VI. Conclusion

In the risk measurement method based on probability semimetric, the new fund performance measurement PM satisfies many fine properties. The most important thing is that it satisfies the random dominant monotonicity of VaR and the standard deviation is not satisfied, which solves the problem of the Sharpe ratio paradox. In addition, it also considers the information of higher moments, so it can give a more reasonable ranking of fund performance. When the distribution is uncertain, a new fund performance measure GPM is defined in the sublinear expectation space. It satisfies the monotonic nature and can rank the fund performance to some extent. It has a certain theoretical significance. It provides some ideas for the future research of scholars.

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