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CAPM Model in the Absence of Risk-Free Rate

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Abstract *: The classic Capital Asset Pricing Model (CAPM) assumes frictionless financial markets with risk-free rates and zero borrowing/lending spreads. However, risk-free rates are not present in actual financial markets. Modern finance economics is faced with a difficult dilemma as a result: how to describe and build the CAPM model without using risk-free rates. Using utility function characteristics, we alter the CAPM model in this study to overcome this problem. Our study is original, and the efficacy and logic of the suggested model are supported by empirical research.*

To optimize utility, we use the Lagrange methodology and tangency method. Calculated ideal weights and risk levels for various assets are based on historical stock annual returns. The tangency method validates the effectiveness and accuracy of the optimal investment strategy. Empirical results show that our utility maximization-based CAPM model outperforms traditional models in terms of investment strategy effectiveness.

Keywords - *CAPM model, Utility function, Optimal investment, Empirical analysis.*

I. Utility Function Introduction And Model Improvement

1.1 CAPM Model

For any portfolio P, its expected return $E(r_p)$ satisfies the following relationships：

$$E(r_p) = r_f + \beta_{pm}[E(r_m) - r_f]$$

Where r_f is the risk-free rate; β_{pm} is the portfolio's sensitivity to market portfolio risk; r_m is the return on portfolio; and $E(r_m) - r_f$ is called the risk premium.

1.2 Introduction to Investor Utility

1. Utility

Utility refers to the degree of satisfaction consumers feel when they consume goods; utility theory is a theory that analyzes how decision-makers approach risk to study consumer behavior. In Western economics, utility theory is divided into two categories, base utility theory and ordinal utility theory; base utility theory believes that utility, like length and weight, can be measured and summed up, and the content of the theory includes the important law of diminishing marginal utility; ordinal utility theory believes that the comparison of utility can only be made through the number of ordinal or rank, and the size of the utility can't be measured, and the theory includes the method of analysis of non-differentiated curves[1].

2. Utility function

Definition: let there exist preferences ≻, ~ on the choice space X, and the function $u: X \to \mathbb{R}$ is said to be a utility function expressing the preferences ≻, ~ ,if for $\forall\ x,y \in X$, there are

$$x \sim y \Leftrightarrow u(x)=u(y)\ ,\ x \succ y \Leftrightarrow u(x)>u(y)$$

In economics, the utility function assigns a numerical value to each bundle of consumption, with larger utility function values assigned to more preferred bundles of consumption, in which case we commonly use ordinal utility theory to understand it; however, the same characteristics of the utility function that assign values to bundles of consumption can also be used for labeling, i.e., for assigning a larger utility function value to undifferentiated curves that lie on top of the coordinate system.

3. Quadratic utility function

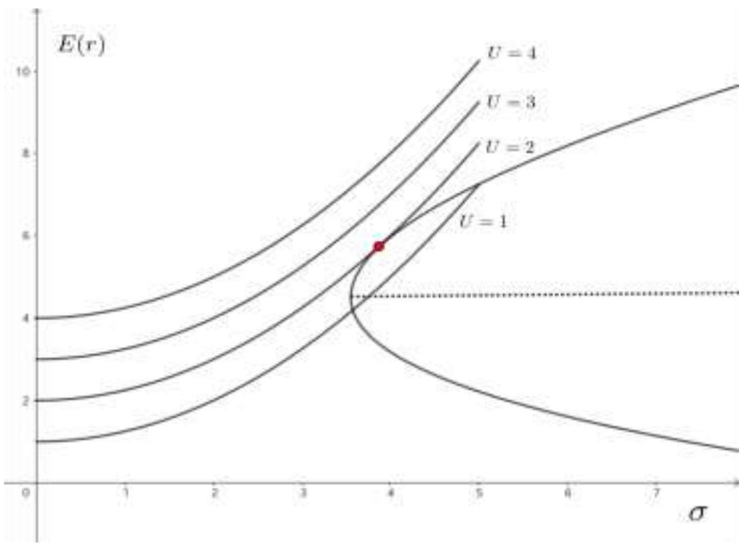
Form: $u(x)=x-bx^2$, $x<\dfrac{1}{2b}$ and $b>0$, where b is the risk aversion coefficient.

The risk-averse utility function has a marginal diminishing effect, that is, the function is concave, with the rate of increase of utility becoming progressively slower as the preference for choice increases.

Zvi Bodi in "Investment Science" has introduced a utility function used to analyze investor behavior in the following form:

$$u(E(r),\sigma^2)=E(r)-\frac{1}{2}A\sigma^2 \qquad ①$$

where $E(r)$ denotes the expectation of the portfolio return, σ^2 is the variance of the portfolio return, and A is used to denote the degree of risk aversion of the investor, with $A>0$ indicating that the



[1] Undifferentiated curve: a curve where the points on the curve correspond to different combinations of goods but correspond to the same degree of utility.

investor is risk averse.

To facilitate understanding, we can consider a set of $(E(r),\sigma^2)$ as two characteristics of a consumption bundle, we can know that the higher the expected return and the smaller the variance, the greater the degree of preference for this consumption bundle (or portfolio), and thus the greater the value of the utility function.

1.3 Maximize expected utility

The theory of investor utility maximization studies under what conditions investors realize utility is great, in this case, investors will choose the portfolio that maximizes the total utility, to achieve the "investor equilibrium", that is to say, any rational investor with the same utility function, in the face of the same kind of portfolio, will choose the portfolio with the largest expected utility. That is to say, any rational investor with the same utility function, when facing the same kind of investment portfolio, will choose the portfolio with the largest expected utility, which also lays a unified foundation for us to introduce the investor's utility into the CAPM model.

The simplest way to realize utility maximization is the image tangent point method, we put the above family of undifferentiated curves and Markowitz effective frontier under the same plane coordinate system, according to the convexity of the undifferentiated curves and the concavity of the effective frontier[2]， the effective frontier must be tangent to one of the family of undifferentiated curves at a point, and utility maximization is reached at this point, in fact, it is geometrically obvious to draw this conclusion.

The image method is certainly intuitive, but when the characteristics of the consumption bundle (portfolio) become more numerous or the utility function is more complex, it is often not possible to draw specific curves, and in the illustration, we use the undifferentiated curves of return-risk mentioned in the previous subsection, and in the next subsection we will utilize such a quadratic utility function to find the point of utility maxima using the Lagrange function method.

1.4 Optimal Asset Portfolio under Utility Functions

We replace $\frac{1}{2}A$ with the risk aversion coefficient b, using the following utility function:

$$u(E(r),\sigma^2)=E(r)-b\sigma^2$$

Consider the M-V model, and assume that there are n assets, the expected return of the i_{th} asset is denoted as r_i,the covariance of the i_{th} and j_{th} assets is denoted as σ_{th} , and the weight corresponding to the i_{th} asset in the portfolio is denoted as $x_i(0\leqslant x_i\leqslant 1)$, here, we abbreviate the expected return $E(r)$ as r , which gives us

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[2] The effective frontier is derived by minimizing the variance model, and its shape in the coordinate system resembles the upper part of a parabola with an opening to the right.

$$r = \sum_{i=1}^{n} r_i x_i$$

$$\sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}$$

If written in the form of a matrix product, remember that the weight vector is $x = (x_1, x_2, \cdots, x_n)'$, the expected return vector is $r = (r_1, r_2, \cdots, r_n)'$, and the covariance matrix is $V = (\sigma_{ij})_{n\times n}$ (there exists an inverse matrix V^{-1}), we have：

$$r = r'x$$
$$\sigma^2 = x'Vx$$

Next, we maximize the expected utility

$$E(u) = E(r - b\sigma^2) = r - b\sigma^2$$

it also translates into the following minimization model:

$$\min \;\; bx'Vx - r'x$$
$$s.t. \;\; x'e = 1$$

Where $e = (1, 1, \cdots, 1)'$, is an n-dimensional unit column vector.

The Lagrange multiplier method is used below to solve for the point where the expected utility is very large, which is the minimum of the above model.

The Lagrange function is:

$$L(x', \lambda) = bx'Vx - r'x - \lambda(x'e - 1)$$

take partial derivatives with respect to L and make them both equal to zero:

$$\frac{\partial L}{\partial x} = 2bVx - r - \lambda e = 0 \qquad (1)$$

$$\frac{\partial L}{\partial \lambda} = x'e - 1 = 0 \qquad (2)$$

both sides of Eq.(1) are simultaneously left–multiplied by $e'V^{-1}$, to give:

$$2be'x - e'V^{-1}r - \lambda e'V^{-1}e = 0$$

and according to Eq.(2), $e'x = (x'e)' = 1$, there is:

$$\lambda = \frac{2b - e'V^{-1}r}{e'V^{-1}e}$$

noting that $R = e'V^{-1}r$, $S = e'V^{-1}e$, notice that S represents the sum of the elements of the matrix V^{-1}.

Bringing the expression for λ into Eq. (1) yields:

$$x^* = \frac{1}{2b}V^{-1}\left(r + \frac{2b - R}{S}e\right)$$

this is the weight vector corresponding to the optimal asset portfolio under this utility function.

1.5 Improved CAPM model under utility function

Based on the weight vector corresponding to the optimal asset portfolio obtained in the previous section, we can introduce the expected return and variance corresponding to the optimal asset portfolio:

$$(r^*, \sigma^{*2}) = (r'x^*, x^{*\prime}Vx^*)$$

We correspond this one number to a coordinate in the M-V model, denoted as point $\tilde{M}$ We know that the higher the expected return, the smaller the variance (risk), and the larger the utility, so it is easy to prove that the point of great utility must be on the Markowitz efficient frontier.

According to the basic assumptions of the CAPM model, we know that all investors who satisfy the assumptions of the CAPM model will choose the market portfolio with the risk-free asset, however, under the utility function expressed in equation ①, these investors will choose the point $\tilde{M}$ as the new "market portfolio", so that each risk aversion coefficient b corresponds to a unique utility function, which also corresponds to a unique utility function, which also corresponds to a unique utility function. utility function, which also corresponds to a unique optimal asset portfolio $\tilde{M}$,obtained by substituting into the classical CAPM model:

$$r_i(b) = r_f + \frac{\sigma_{i\tilde{M}}}{\sigma^{*2}}(r^* - r_f)$$

Where $\sigma_{i\tilde{M}}$ cannot be obtained by theoretical calculations and needs to be estimated by specific statistics.

II. An empirical investigation of optimal investment strategies under expected utility maximization

2.1 Data selection and pre-processing

Five real estate stocks are selected as the five risky assets, and a total of 20 years of annual stock returns from 2003-2022 are collected as the raw data, and we calculate the average returns of the five risky assets as shown in the table below:

Table 1 Average annual return of five stocks

Asset name	Asset1	Asset2	Asset3	Asset4	Asset5
Average rate of return	0.1235	0.1239	0.1208	0.1209	0.1666

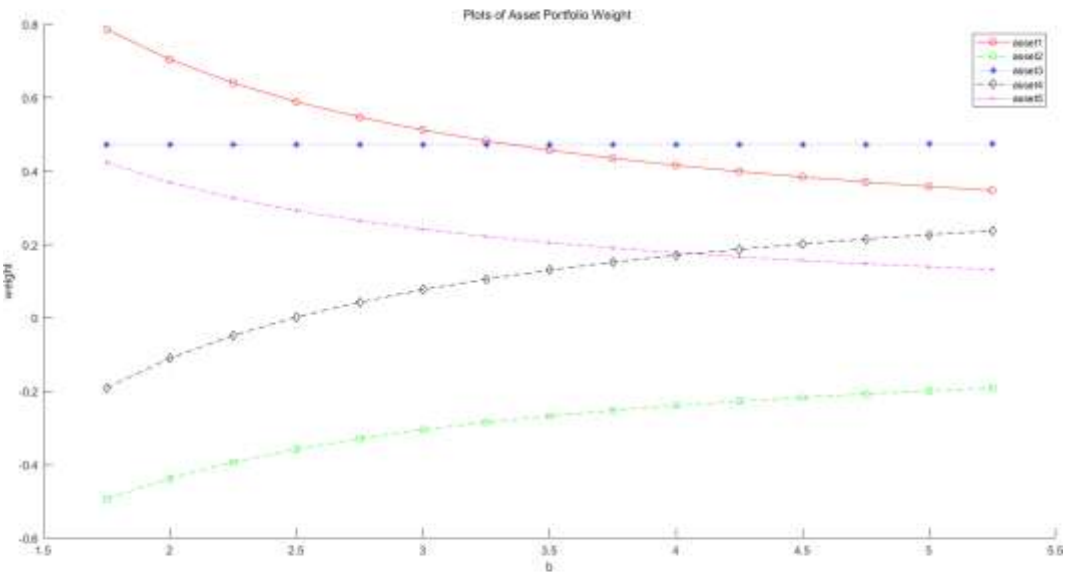
The covariance matrix of the three assets is

$$\begin{pmatrix} 0.00169999 & 0.00054892 & 0.00002768 & 0.00022322 & 0.00037089 \\ 0.00054892 & 0.00178964 & 0.00029484 & 0.00048845 & 0.00402714 \\ 0.00002768 & 0.00029484 & 0.00096129 & -0.00031623 & 0.00004337 \\ 0.00022322 & 0.00048845 & -0.00031623 & 0.00104601 & 0.00198725 \\ 0.00037089 & 0.00402714 & 0.00004337 & 0.00198725 & 0.0366422 \end{pmatrix}$$

2.2 Analysis of results

According to the optimal portfolio weight vector formula derived above, we calculate the optimal investment ratio under different risk aversion coefficients b as well as the corresponding $\widetilde{M}$ points and utility maxima, respectively.

To facilitate the observation of the change pattern of the optimal investment ratio with the risk aversion coefficient, we plot the trend of each stock weight as follows:



Notes: Here we briefly explain the concept of negative weighting, when calculating the proportion of assets in the portfolio, if the weight is negative, it means that the corresponding asset is not worth investing in under the comprehensive consideration of the return and risk, and the investor often chooses to sell the asset short, so as to reduce a part of the risk, and then the proceeds of the short-sale are used to invest in the other assets in the portfolio with a larger weight, to obtain the indirect benefit; in particular, if the weight of An asset with a weight of more than 1 indicates that the asset has a high return and low risk and is the main influence on increasing utility, there must also be at least one negatively-weighted asset, i.e., an asset that needs to be sold short, to finance the purchase of this asset.

By observing and analyzing the trend of weight changes, we get:

(1) The weight of stock 2 is always negative, indicating that the investor needs to sell the asset to reduce the risk under this utility function.

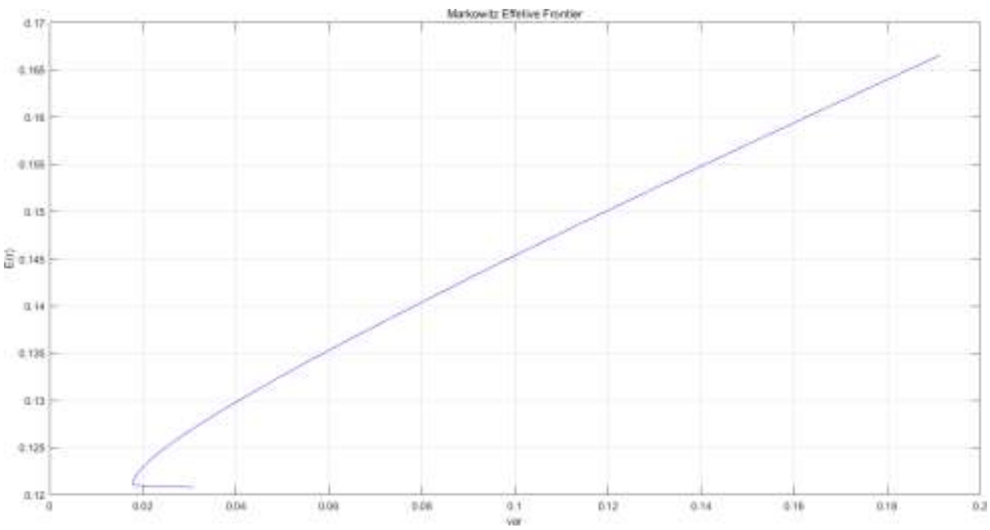
(2) The weight of stock 3 is stable at 0.473, indicating that stock 3 is less risky and has little

correlation with other stocks.

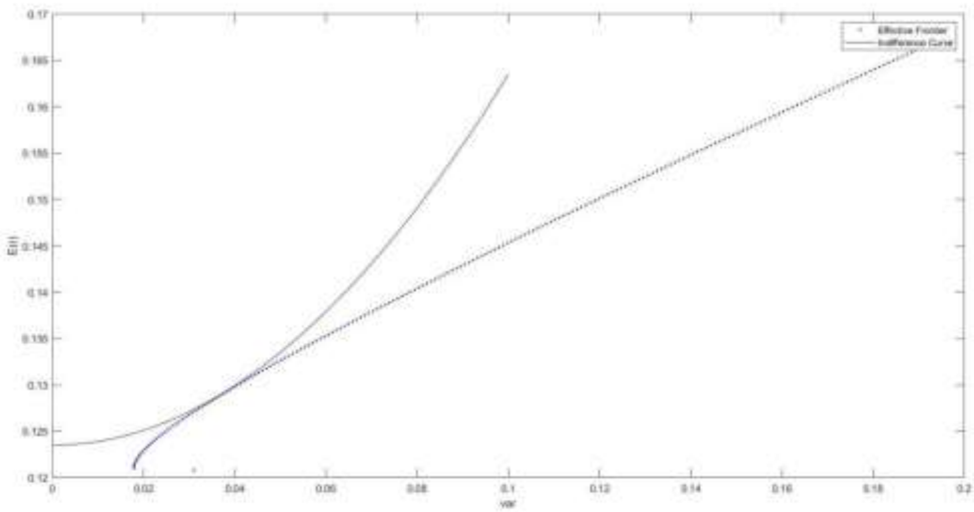
(3) As the risk aversion coefficient increases, investors become more risk averse, and the weight of stock 4 turns from negative to positive, which indicates that investors are risk averse. After reaching a certain level, it will change from selling the stock to buying the stock as a source of income.

(4) As investors become more risk averse, the investment ratio of each stock stabilizes, which is due to the fact that the total risk affects investors' utility more and more. This is because the total risk affects the investor's utility more and more, so the stocks are gradually diversified, which is also consistent with the law of diversification to reduce the risk of investors.

Then we use the optimization function algorithm to plot the Markowitz efficient frontier corresponding to these five stocks, as shown in Fig:



Then we use the cut-point method to verify the optimal portfolios obtained. Since the Markowitz efficient frontier has to ensure that the weights cannot be negative, the set of portfolios we examine contains all the portfolios on the Markowitz efficient frontier, and we choose the risk aversion coefficient $b = 4.00$ as an example. We put the Markowitz efficient frontier and the one-family utility function non-discrepancy curve with $b = 4.00$ in the same $r - \sigma^2$ coordinate system and adjust the intercept (utility value) so that the undifferentiated curve is tangent to the efficient frontier, as shown in Fig.



The intercept (utility value) of the undifferentiated curve at tangency is calculated to be 0.12358, and the corresponding expected return-variance coordinate is:

$$(E(r),\sigma^2)=(0.12941,0.03838)$$

Checking Table 2 shows that the value of the utility function is 0.12366, the expected return is 0.12934, and the variance is 0.00142. Since the variances of the optimal portfolios are all very small, the expected return is the main factor affecting the utility function, so the optimal asset portfolios found by the two methods are very close to each other in terms of the expected return and the value of the utility function; however, the variance of the optimal asset portfolios computed by the minimization model is 0.37 times that of the optimal point on the efficient frontier, which also indicates that adding negative weights to the set of corresponding asset portfolios and using short selling measures for some assets does reduce risk and improve investor utility.

III. Conclusion

The optimal portfolios calculated by the expected utility maximization model for the five real estate stocks are reasonable and efficient, not only intuitively consistent with investors' sensitivity to risk return but also theoretically consistent with the basic assumptions of risk-averse investors. Combined with the utility function we used and the optimal portfolio found by the cut-point method, we confirm that the results of the maximization model do achieve great utility, and under the condition of allowing short selling, it makes the utility value not decrease while reducing the risk; if short selling is not considered, the investment value of the assets under study can also be clearly judged, so as to reasonably choose the investment ratio and funding strategy.

The results of this study show that the utility function we use can effectively find the optimal investment strategy, which can be applied in asset management, venture capital, and other related fields, while the theoretical support of the model is in line with the basic assumptions of the CAPM classical model, which also brings certain scientific value to the pricing of capital assets in the financial market.

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